A PRIMER OF REAL ANALYSIS

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A Primer of Real Analysis

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Licensing

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CHAPTER OVERVIEW

1: Fundamentals

1.1: Sets and Relations
1.2: Functions
1.3: Rational Numbers
1.4: Real Numbers

Thumbnail: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$. (CC BY; OpenStax)

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1.1: Sets and Relations

1.1.1 The Integers

Kronecker once said, "God made the integers; all the rest is the work of man." Taking this as our starting point, we assume the existence of the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}. \tag{1.1.1}$$

the set of integers. Moreover, we assume the properties of the operations of addition and multiplication of integers, along with other elementary properties such as the Fundamental Theorem of Arithmetic, that is, the statement that every integer may be factored into a product of prime numbers and this factorization is essentially unique.

1.1.2 Sets

We will take a naive view of sets: given any property p, we may determine a set by collecting together all objects which have property p. This may be done either by explicit enumeration, such as, p is the property of being one of a, b, or c, which creates the set $\{a, b, c\}$, or by stating the desired property, such as, p is the property of being a positive integer, which creates the set

$$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}. \tag{1.1.2}$$

The notation $x \in A$ indicates that x is an element of the set A. Given sets A and B, we say A is a subset of B, denoted $A \subset B$, if from the fact that $x \in A$ it necessarily follows that $x \in B$. We say sets A and B are equal if both $A \subset B$ and $B \subset A$.

Given two sets A and B, we call the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \tag{1.1.3}$$

the union of A and B and the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{1.1.4}$$

the intersection of A and B. We call the set

$$A \backslash B = \{x : x \in A, x \notin B\} \tag{1.1.5}$$

the difference of A and B.

More generally, if I is a set and $\{A_\alpha : \alpha \in I\}$ is a collection of sets, one for each element of I, then we have the union

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha\} \tag{1.1.6}$$

and the intersection

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha\}. \tag{1.1.7}$$

Example 1.1.1

For example, if $I = \{2, 3, 4, \ldots\}$ and we let

$$A_2 = \{n : n \in \mathbb{Z}^+, n > 1, n \neq 2m \text{ for any } m \in \mathbb{Z}^+ \text{with } m > 1\} \tag{1.1.8}$$

and, for any $i \in I, i > 2$,

$$A_i = \{n : n \in A_{i-1}, n \neq mi \text{ for any } m \in \mathbb{Z}^+ \text{with } m > 1\}, \tag{1.1.9}$$

then $\cap_{i \in I} A_i$ is the set of prime numbers.

If A and B are both sets, we call the set

$$A \times B = \{(a, b) : a \in A, b \in B\} \tag{1.1.10}$$

the cartesian product of A and B. If $A = B$, we write

$$A^2 = A \times A. \tag{1.1.11}$$

Example 1.1.2

$\mathbb{Z}^2 = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}\}$ is the set of all ordered pairs of integers.

1.1.3 Relations

Given two sets A and B, we call a subset R of $A \times B$ a relation. Given a relation R, we will write $a \sim_R b$, or simply $a \sim b$ if R is clear from the context, to indicate that $(a, b) \in R$.

Example 1.1.3

We say that an integer m divides and integer n if $n = mi$ for some integer i. If we let

$$R = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}, m \text{ divides } n\}, \tag{1.1.12}$$

then R is a relation. For example, $3 \sim_R 12$.

Consider a set A and a relation $R \subset A^2$. For purposes of conciseness, we say simply that R is a relation on A. If R is such that $a \sim_R a$ for every $a \in A$, we say R is reflexive; if R is such that $b \sim Ra$ whenever $a \sim_R b$, we say R is symmetric; if $a \sim_R b$ and $b \sim_R c$ together imply $a \sim_R c$, we say R is transitive. We call a relation which is reflexive, symmetric, and transitive an equivalence relation.

Exercise 1.1.1

Show that the relation R on $\mathbb{Z}$ defined by $m \sim_R n$ if m divides n is reflexive and transitive, but not symmetric.

Exercise 1.1.2

Show that the relation R on $\mathbb{Z}$ defined by $m \sim_R n$ if $m - n$ is even is an equivalence relation.

Given an equivalence relation R on a set A and an element $x \in A$, we call

$$[x] = \{y : y \in A, y \sim_R x\} \tag{1.1.13}$$

the equivalence class of x.

Exercise 1.1.3

Given an equivalence relation R on a set A, show that

a. $[x] \cap [y] \neq \emptyset$ if and only if $x \sim_R y$

b. $[x] = [y]$ if and only if $x \sim_R y$.

As a consequence of the previous exercise, the equivalence classes of an equivalence relation on a set A constitute a partition of A (that is, A may be written as the disjoint union of the equivalence classes).

Exercise 1.1.4

Find the equivalence classes for the equivalence relation in Exercise 1.1.2.

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1.2: Functions

If A and B are sets, we call a relation $R \subset A \times B$ a function with domain A if for every $a \in A$ there exists one, and only one, $b \in B$ such that $(a, b) \in R$. We typically indicate such a relation with the notation $f : A \to B$, and write $f(a) = b$ to indicate that $(a, b) \in R$. We call the set of all $b \in B$ such that $f(a) = b$ for some $a \in A$ the range of f. With this notation, we often refer to R as the graph of f.

We say $f : A \to B$ is one-to-one if for every b in the range of f there exists a unique $a \in A$ such that $f(a) = b$. We say f is onto if for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$. For example, the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(z) = z^2$ is one-to-one, but not onto, whereas the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z + 1$ is both one-to-one and onto.

Given two functions, $g : A \to B$ and $f : B \to C$, we define the composition, denoted $f \circ g : A \to C$, to be the function defined by $f \circ g(a) = f(g(a))$.

If $f : A \to B$ is both one-to-one and onto, then we may define a function $f^{-1} : B \to A$ by requiring $f^{-1}(b) = a$ if and only if $f(a) = b$. Note that this implies that $f \circ f^{-1}(b) = b$ for all $b \in B$ and $f^{-1} \circ f(a) = a$ for all $a \in A$. We call f^{-1} the inverse of f.

Given any collection of nonempty sets, $\{A_\alpha\}, \alpha \in I$, we assume the existence of a function $\phi : I \to B = \bigcup_{\alpha \in I} A_\alpha$, with the property that $\phi(\alpha) \in A_\alpha$. We call such a function a choice function. The assumption that choice functions always exist is known as the Axiom of Choice.

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1.3: Rational Numbers

Let $P = \{(p, q) : p, q \in \mathbb{Z}, q \neq 0\}$. We define an equivalence relation on P by saying $(p, q) \sim (s, t)$ if $pt = qs$.

? Exercise 1.3.1

Show that the relation as just defined is indeed an equivalence relation.

We will denote the equivalence class of $(p, q) \in P$ by p/q, or $\frac{p}{q}$. We call the set of all equivalence classes of P the rational numbers, which we denote by $\mathbb{Q}$. If $p \in \mathbb{Z}$, we will denote the equivalence class of $(p, 1)$ by p; that is, we let

$$\frac{p}{1} = p. \tag{1.3.1}$$

In this way, we may think of $\mathbb{Z}$ as a subset of $\mathbb{Q}$.

1.3.1 Field Properties

We wish to define operations of addition and multiplication on elements of $\mathbb{Q}$. We begin by defining operations on the elements of P. Namely, given $(p, q) \in P$ and $(s, t) \in P$, define

$$(p, q) \oplus (s, t) = (pt + sq, qt) \tag{1.3.2}$$

and

$$(p, q) \otimes (s, t) = (ps, qt). \tag{1.3.3}$$

Now suppose $(p, q) \sim (a, b)$ and $(s, t) \sim (c, d)$. It follows that $(p, q) \oplus (s, t) \sim (a, b) \oplus (c, d)$, that is, $(pt + sq, qt) \sim (ad + cb, bd)$, since

$$(pt + sq)bd = pbtd + sdqb = qatd + tcqb = (ad + cb)qt. \tag{1.3.4}$$

Moreover, $(p, q) \otimes (s, t) \sim (a, b) \otimes (c, d)$, that is, $(ps, qt) \sim (ac, bd)$, since

$$psbd = pbsd = qatc = qtac. \tag{1.3.5}$$

This shows that the equivalence class of a sum or product depends only on the equivalence classes of the elements being added or multiplied. Thus we may define addition and multiplication on $\mathbb{Q}$ by

$$\frac{p}{q} + \frac{s}{t} = \frac{pt + sq}{qt} \tag{1.3.6}$$

and

$$\frac{p}{q} \times \frac{s}{t} = \frac{ps}{qt}, \tag{1.3.7}$$

and the results will not depend on which representatives we choose for each equivalence class. Of course, multiplication is often denoted using juxtaposition, that is,

$$\frac{p}{q} \times \frac{s}{t} = \frac{p}{q}\frac{s}{t}, \tag{1.3.8}$$

and repeated multiplication may be denoted by exponentiation, that is, a^n, $a \in \mathbb{Q}$ and $n \in \mathbb{Z}^+$, represents the product of a with itself n times.

Note that if $(p, q) \in P$, then $(-p, q) \sim (p, -q)$. Hence, if $a = \frac{p}{q} \in \mathbb{Q}$, then we let

$$-a = \frac{-p}{q} = \frac{p}{-q}. \tag{1.3.9}$$

For any $a, b \in \mathbb{Q}$, we will write $a - b$ to denote $a + (-b)$.

If $a = \frac{p}{q} \in \mathbb{Q}$ with $p \neq 0$, then we let

$$a^{-1} = \frac{q}{p}. \tag{1.3.10}$$

Moreover, we will write

$$\frac{1}{a} = a^{-1}, \tag{1.3.11}$$

$$\frac{1}{a^n} = a^{-n} \tag{1.3.12}$$

for any $n \in \mathbb{Z}^+$, and, for any $b \in \mathbb{Q}$,

$$\frac{b}{a} = ba^{-1}. \tag{1.3.13}$$

It is now easy to show that

1. $a + b = b + a$ for all $a, b \in \mathbb{Q}$;
2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Q}$;
3. $ab = ba$ for all $a, b \in \mathbb{Q}$;
4. $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{Q}$;
5. $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{Q}$;
6. $a + 0 = a$ for all $a \in \mathbb{Q}$;
7. $a + (-a) = 0$ for all $a \in \mathbb{Q}$;
8. $1a = a$ for all $a \in \mathbb{Q}$;
9. if $a \in \mathbb{Q}, a \neq 0$, then $aa^{-1} = 1$.

Taken together, these statements imply that $\mathbb{Q}$ is a field.

1.3.2 Order and Metric Properties

We say a rational number a is positive if there exist $p, q \in \mathbb{Z}^+$ such that $a = \frac{p}{q}$. We denote the set of all positive elements of $\mathbb{Q}$ by $\mathbb{Q}^+$.

Given $a, b \in \mathbb{Q}$, we say a is less than b, or, equivalently, b is greater than a, denoted either by $a < b$ or $b > a$, if $b - a$ is positive. In particular, $a > 0$ if and only if a is positive. If $a < 0$, we say a is negative. We write $a \leq b$, or, equivalently, $b \geq a$, if either $a < b$ or $a = b$.

? Exercise 1.3.2

Show that for any $a \in \mathbb{Q}$, one and only one of the following must hold: (a) $a < 0, (b) a = 0, (c) a > 0$.

? Exercise 1.3.3

Show that if $a, b \in \mathbb{Q}^+$, then $a + b \in \mathbb{Q}^+$.

? Exercise 1.3.4

Suppose $a, b, c \in \mathbb{Q}$. Show each of the following:

a. One, and only one, of the following must hold:

(i) $a < b$,

(ii) $a = b$,

(iii) $a > b$.

b. If $a < b$ and $b < c$, then $a < c$.

c. If $a < b$, then $a + c < b + c$.

d. If $a > 0$ and $b > 0$, then $ab > 0$.

Exercise 1.3.5

Show that if $a, b \in \mathbb{Q}$ with $a > 0$ and $b < 0$, then $ab < 0$.

Exercise 1.3.6

Show that if $a, b, c \in \mathbb{Q}$ with $a < b$, then $ac < bc$ if $c > 0$ and $ac > bc$ if $c < 0$.

Exercise 1.3.7

Show that if $a, b \in \mathbb{Q}$ with $a < b$, then

$$a < \frac{a+b}{2} < b. \tag{1.3.14}$$

As a consequence of Exercise 1.3 .4 we say $\mathbb{Q}$ is an ordered field. For any $a \in \mathbb{Q}$, we call

$$|a| = \begin{cases} a, \text{ if } a \geq 0, \\ -a, \text{ if } a < 0, \end{cases} \tag{1.3.15}$$

the absolute value of a.

Exercise 1.3.8

Show that for any $a \in \mathbb{Q}, -|a| \leq a \leq |a|$.

Proposition 1.3.1

For any $a, b \in \mathbb{Q}, |a+b| \leq |a| + |b|$.

Proof

If $a + b \geq 0$, then

$$|a| + |b| - |a+b| = |a| + |b| - a - b = (|a| - a) + (|b| - b). \tag{1.3.16}$$

Both of the terms on the right are nonnegative by Exercise 1.3.8. Hence the sum is nonnegative and the proposition follows. If $a + b < 0$, then

$$|a| + |b| - |a+b| = |a| + |b| + a + b = (|a| + a) + (|b| + b). \tag{1.3.17}$$

Again, both of the terms on the right are nonnegative by Exercise 1.3.8. Hence the sum is nonnegative and the theorem follows. Q.E.D.

It is now easy to show that the absolute value satisfies

1. $|a - b| \geq 0$ for all $a, b \in \mathbb{Q}$, with $|a - b| = 0$ if and only if $a = b$,
2. $|a - b| = |b - a|$ for all $a, b \in \mathbb{Q}$,
3. $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbb{Q}$.

Note that the last statement, known as the triangle inequality, follows from writing

$$a - b = (a - c) + (c - b) \tag{1.3.18}$$

and applying the previous proposition. These properties show that the function

$$d(a, b) = |a - b| \tag{1.3.19}$$

is a metric, and we will call $|a - b|$ the distance from a to b.

Suppose $a, b \in \mathbb{Q}^+$ with $a < b$ and let $p, q, r, s \in \mathbb{Z}^+$ such that $a = \frac{p}{q}$ and $b = \frac{r}{s}$. For any $n \in \mathbb{Z}^+$, we have

$$na - b = n\frac{p}{q} - \frac{r}{s} = \frac{nps - rq}{qs}. \tag{1.3.20}$$

If we choose n large enough so that $nps - rq > 0$, it follows that $na - b > 0$, that is, $na > b$. We say that the ordered field $\mathbb{Q}$ is archimedean. Note that it also follows that we may choose n large enough to ensure that $\frac{b}{n} < a$.

1.3.3 Upper and Lower Bounds

Definition

Let $A \subset \mathbb{Q}$. If $s \in \mathbb{Q}$ is such that $s \geq a$ for every $a \in A$, then we call s an upper bound for A. If s is an upper bound for A with the property that $s \leq t$ whenever t is an upper bound for A, then we call s the supremum, or least upper bound, of A, denoted $s = \sup A$. Similarly, if $r \in \mathbb{Q}$ is such that $r \leq a$ for every $a \in A$, then we call r a lower bound for A. If r is a lower bound for A with the property that $r \geq t$ whenever t is a lower bound for A, then we call r the infimum, or greatest lower bound, of A, denoted $r = \inf A$.

Exercise 1.3.9

Show that the supremum of a set $A \subset \mathbb{Q}$, if it exists, is unique, and thus justify the use of the definite article in the previous definition.

A set which does not have an upper bound will not, *a fortiori*, have a supremum. Moreover, even sets which have upper bounds need not have a supremum.

Example 1.3.1

$\mathbb{Q}$ does not have an upper bound.

Example 1.3.2

Consider the set

$$A = \left\{a : a \in \mathbb{Q}^+, a^2 < 2\right\}. \tag{1.3.21}$$

Note that if $a, b \in \mathbb{Q}^+$ with $a < b$, then

$$b^2 - a^2 = (b - a)(b + a) > 0, \tag{1.3.22}$$

from which it follows that $a^2 < b^2$. Hence if $a \in \mathbb{Q}^+$ with $a^2 > 2$, then a is an upper bound for A. For example, 4 is an upper bound for A.

Now suppose $s \in \mathbb{Q}^+$ is the supremum of A. We must have either $s^2 < 2$, $s^2 > 2$, or $s^2 = 2$.

Suppose $s^2 < 2$ and let $\epsilon = 2 - s^2$. By the archimedean property of $\mathbb{Q}$, we may choose $n \in \mathbb{Z}^+$ such that

$$\frac{2s + 1}{n} < \epsilon, \tag{1.3.23}$$

from which it follows that

$$\frac{2s}{n} + \frac{1}{n^2} = \frac{2s + \frac{1}{n}}{n} \leq \frac{2s + 1}{n} < \epsilon. \tag{1.3.24}$$

Hence

$$\left(s + \frac{1}{n}\right)^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2} < s^2 + \epsilon = 2, \tag{1.3.25}$$

which implies that $s + \frac{1}{n} \in A$. since $s < s + \frac{1}{n}$, this contradicts the assumption that s is an upper bound for A.

So now suppose $s^2 > 2$. Again let $n \in \mathbb{Z}^+$ and note that

$$\left(s - \frac{1}{n}\right)^2 = s^2 - \frac{2s}{n} + \frac{1}{n^2}. \tag{1.3.26}$$

If we let $\epsilon = s^2 - 2$, then we may choose $n \in \mathbb{Z}^+$ so that

$$\frac{2s}{n} < \epsilon. \tag{1.3.27}$$

It follows that

$$\left(s - \frac{1}{n}\right)^2 > s^2 - \epsilon + \frac{1}{n^2} = 2 + \frac{1}{n^2} > 2. \tag{1.3.28}$$

Thus $s - \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n} < s$, contradicting the assumption that $s = \sup A$.

Thus we must have $s^2 = 2$. However, this is impossible in light of the following proposition. Hence we must conclude that A does not have a supremum.

Proposition 1.3.2

There does not exist a rational number s with the property that $s^2 = 2$.

Proof

Suppose there exists $s \in \mathbb{Q}$ such that $s^2 = 2$. Choose $a, b \in \mathbb{Z}^+$ so that a and b are relatively prime (that is, they have no factor other than 1 in common) and $s = \frac{a}{b}$. Then

$$\frac{a^2}{b^2} = 2, \tag{1.3.29}$$

so $a^2 = 2b^2$. Thus a^2, and hence a, is an even integer. So there exists $c \in \mathbb{Z}^+$ such that $a = 2c$. Hence

$$2b^2 = a^2 = 4c^2, \tag{1.3.30}$$

from which it follows that $b^2 = 2c$, and so b is also an even integer. But this contradicts the assumption that a and b are relatively prime. Q.E.D.

Exercise 1.3.10

Show that there does not exist a rational number s with the property that $s^2 = 3$.

Exercise 1.3.11

Show that there does not exist a rational number s with the property that $s^2 = 6$.

Exercise 1.3.12

Let $A = \{a : a \in \mathbb{Q}, a^3 < 2\}$.

1. Show that if $a \in A$ and $b < a$, then $b \in A$.

2. Show that if $a \notin A$, and $b > a$, then $b \notin A$.

1.3.4 Sequences

Definition

Suppose $n \in \mathbb{Z}, I = \{n, n+1, n+2, ...\}$, and A is a set. We call a function $\varphi : I \to A$ a sequence with values in A.

Frequently, we will define a sequence φ by specifying its values with notation such as, for example, $\{\varphi(i)\}_{i \in I}$, or $\{\varphi(i)\}_{i=n}^{\infty}$. Thus, for example, $\{i^2\}_{i=1}^{\infty}$ denotes the sequence $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ defined by $\varphi(i) = i^2$. Moreover, it is customary to denote the values of a sequence using subscript notation. Thus if $a_i = \varphi(i)$, $i \in I$, then $\{a_i\}_{i \in I}$ denotes the sequence φ. For example, we may define the sequence of the previous example by writing $a_i = i^2, i = 1, 2, 3, ...$

Definition

Suppose $\{a_i\}_{i\in I}$ is a sequence with values in $\mathbb{Q}$. We say that $\{a_i\}_{i\in I}$ converges, and has limit $L, L \in \mathbb{Q}$, if for every $\epsilon \in \mathbb{Q}^+$, there exists $N \in \mathbb{Z}$ such that

$$|a_i - L| < \epsilon \text{ whenever } i > N. \tag{1.3.31}$$

If the sequence $\{a_i\}_{i\in I}$ converges to L, we write

$$\lim_{i\to\infty} a_i = L. \tag{1.3.32}$$

Example 1.3.3

We have

$$\lim_{i\to\infty} \frac{1}{i} = 0, \tag{1.3.33}$$

since, for any rational number $\epsilon > 0$,

$$\left|\frac{1}{i} - 0\right| = \frac{1}{i} < \epsilon \tag{1.3.34}$$

for any $i > N$, where N is any integer larger than $\frac{1}{\epsilon}$.

Definition

Suppose $\{a_i\}_{i\in I}$ is a sequence with values in $\mathbb{Q}$. We call $\{a_i\}_{i\in I}$ a Cauchy sequence if for every $\epsilon \in \mathbb{Q}^+$, there exists $N \in \mathbb{Z}$ such that

$$|a_i - a_k| < \epsilon \text{ whenever both } i > N \text{ and } k > N. \tag{1.3.35}$$

Proposition 1.3.3

If $\{a_i\}_{i\in I}$ converges, then $\{a_i\}_{i\in I}$ is a Cauchy sequence.

Proof

Suppose $\lim_{i\to\infty} a_i = L$. Given $\epsilon \in \mathbb{Q}^+$, choose an integer N such that

$$|a_i - L| < \frac{\epsilon}{2} \tag{1.3.36}$$

for all $i > N$. Then for any $i, k > N$, we have

$$|a_i - a_k| = |(a_i - L) + (L - a_k)| \le |a_i - L| + |a_k - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.3.37}$$

Hence $\{a_i\}_{i\in I}$ is a Cauchy sequence. Q.E.D.

The proposition shows that every convergent sequence in $\mathbb{Q}$ is a Cauchy sequence, but, as the next example shows, the converse does not hold.

Example 1.3.1

Let

$$f(x) = x^2 - 2 \tag{1.3.38}$$

and consider the sequence constructed as follows: Begin by setting $a_1 = 1$, $b_1 = 2$, and $x_1 = \frac{3}{2}$. If $f(a_1) f(x_1) < 0$, set

$$x_2 = \frac{a_1 + x_1}{2}, \tag{1.3.39}$$

$a_2 = a_1$, and $b_2 = x_1$; otherwise, set

$$x_2 = \frac{x_1 + b_1}{2}, \tag{1.3.40}$$

$a_2 = x_1$, and $b_2 = b_1$. In general, given a_n, x_n, and b_n, if $f(a_n) f(x_n) < 0$, set

$$x_{n+1} = \frac{a_n + x_n}{2}, \tag{1.3.41}$$

$a_{n+1} = a_n$, and $b_{n+1} = x_n$; otherwise, set

$$x_{n+1} = \frac{x_n + b_n}{2}, \tag{1.3.42}$$

$a_{n+1} = x_n$, and $b_{n+1} = b_n$. Note that for any positive integer N, $f(a_N) < 0$, $f(b_N) > 0$, and

$$a_N < x_i < b_N \tag{1.3.43}$$

for all $i > N$. Moreover,

$$|b_N - a_N| = \frac{1}{2^{N-1}}, \tag{1.3.44}$$

so

$$|x_i - x_k| < \frac{1}{2^{N-1}} \tag{1.3.45}$$

for all $i, k > N$. Hence given any $\epsilon \in \mathbb{Q}^+$, if we choose an integer N such that $2^{N-1} > \frac{1}{\epsilon}$, then

$$|x_i - x_k| < \frac{1}{2^{N-1}} < \epsilon \tag{1.3.46}$$

for all $i, k > N$, showing that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Now suppose $\{x_i\}_{i=1}^{\infty}$ converges to $s \in \mathbb{Q}$. Note hat we must have

$$a_i \leq s \leq b_i \tag{1.3.47}$$

for all $i \in \mathbb{Z}^+$. If $f(s) < 0$, then, since the set $\{a : a \in \mathbb{Q}^+, a^2 < 2\}$ does not have a supremum, there exists $t \in \mathbb{Q}^+$ such that $s < t$ and $f(t) < 0$. If we choose N so that

$$\frac{1}{2^{N-1}} < t - s, \tag{1.3.48}$$

then

$$|s - b_N| \leq |a_N - b_N| = \frac{1}{2^{N-1}} < t - s. \tag{1.3.49}$$

Hence $b_N < t$, which implies that $f(b_N) < 0$, contradicting the construction of $\{b_i\}_{i=1}^{\infty}$. Hence we must have $f(s) > 0$. But if $f(s) > 0$, then there exists $t \in \mathbb{Q}^+$ such that $t < s$ and $f(t) > 0$. We can then choose N so that $t < a_N$, implying that $f(a_N) > 0$, contradicting the construction of $\{a_i\}_{i=1}^{\infty}$. Hence we must have $f(s) = 0$, which is not possible since $s \in \mathbb{Q}$. Thus we must conclude that $\{x_i\}_{i=1}^{\infty}$ does not converge.

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1.4: Real Numbers

Let C be the set of all Cauchy sequences of rational numbers. We define a relation on C as follows: If $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are Cauchy sequences in $\mathbb{Q}$, then $\{a_i\}_{i\in I} \sim \{b_j\}_{j\in J}$, which we will write more simply as $a_i \sim b_i$, if for every rational number $\epsilon > 0$, there exists an integer N such that

$$|a_i - b_i| < \epsilon \tag{1.4.1}$$

whenever $i > N$. This relation is clearly reflexive and symmetric. To show that it is also transitive, and hence an equivalence relation, suppose $a_i \sim b_i$ and $b_i \sim c_i$. Given $\epsilon \in \mathbb{Q}^+$, choose N so that

$$|a_i - b_i| < \frac{\epsilon}{2} \tag{1.4.2}$$

for all $i > N$ and M so that

$$|b_i - c_i| < \frac{\epsilon}{2} \tag{1.4.3}$$

for all $i > M$. Let L be the larger of N and M. Then, for all $i > L$,

$$|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.4.4}$$

Hence $a_i \sim c_i$.

Definition: set of equivalence classes

Using the equivalence relation just defined, we call the **set of equivalence classes** of C the real numbers, denoted $\mathbb{R}$.

Note that if $a \in \mathbb{Q}$, we may identify a with the equivalence class of the sequence $\{b_i\}_{i=1}^{\infty}$ where $b_i = a, i = 1, 2, 3, \ldots$, and thus consider $\mathbb{Q}$ to be a subset of $\mathbb{R}$.

Exercise 1.4.1

Suppose $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in J}$ are sequences in $\mathbb{Q}$ with

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i. \tag{1.4.5}$$

Show that $a_i \sim b_i$.

1.4.1 Field Properties

Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are both Cauchy sequences of rational numbers. Let $K = I \cap J$ and define a new sequence $\{s_k\}_{k\in K}$ by setting $s_k = a_k + b_k$. Given any rational $\epsilon > 0$, choose integers N and M such that

$$|a_i - a_j| < \frac{\epsilon}{2} \tag{1.4.6}$$

for all $i, j > N$ and

$$|b_i - b_j| < \frac{\epsilon}{2} \tag{1.4.7}$$

for all $i, j > M$. If L is the larger of N and M, then, for all $i, j > L$,

$$|s_i - s_j| = |(a_i - a_j) + (b_i - b_j)| \leq |a_i - a_j| + |b_i - b_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{1.4.8}$$

showing that $\{s_i\}_{k\in K}$ is also a Cauchy sequence. Moreover, suppose $a_i \sim c_i$ and $b_i \sim d_i$. Given $\epsilon \in \mathbb{Q}^+$, choose N so that

$$|a_i - c_i| < \frac{\epsilon}{2} \tag{1.4.9}$$

for all $i > N$ and choose M so that

$$|b_i - d_i| < \frac{\epsilon}{2} \tag{1.4.10}$$

for all $i > M$. If L is the larger of N and M, then, for all $i > L$,

$$|(a_i + b_i) - (c_i + d_i)| \leq |a_i - c_i| + |b_i - d_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.4.11}$$

Hence $a_i + b_i \sim c_i + d_i$. Thus if $u, v \in \mathbb{R}$, with u being the equivalence class of $\{a_i\}_{i \in I}$ and v being the equivalence class of $\{b_j\}_{j \in J}$, then we may unambiguously define $u + v$ to be the equivalence class of $\{a_i + b_i\}_{i \in K}$, where $K = I \cap J$.

Suppose $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are both Cauchy sequences of rational numbers.

Let $K = I \cap J$ and define a new sequence $\{p_k\}_{k \in K}$ by setting $p_k = a_k b_k$. Let $B > 0$ be an upper bound for the set $\{|a_i| : i \in I\} \cup \{|b_j| : j \in J\}$. Given $\epsilon > 0$ choose integers N and M such that

$$|a_i - a_j| < \frac{\epsilon}{2B} \tag{1.4.12}$$

for all $i, j > N$ and

$$|b_i - b_j| < \frac{\epsilon}{2B} \tag{1.4.13}$$

for all $i, j > M$. If L is the larger of N and M, then, for all $i, j > L$,

$$\begin{aligned} |p_i - p_j| &= |a_i b_i - a_j b_j| \\ &= |a_i b_i - a_j b_i + a_j b_i - a_j b_j| \\ &= |b_i (a_i - a_j) + a_j (b_i - b_j)| \\ &\leq |b_i (a_i - a_j)| + |a_j (b_i - b_j)| \\ &= |b_i| |a_i - a_j| + |a_j| |b_i - b_j| \\ &< B\frac{\epsilon}{2B} + B\frac{\epsilon}{2B} \\ &= \epsilon \end{aligned} \tag{1.4.14}$$

Hence $\{p_k\}_{k \in K}$ is a Cauchy sequence.

Now suppose $\{c_i\}_{i \in H}$ and $\{d_i\}_{i \in G}$ are Cauchy sequences with $a_i \sim c_i$ and $b_i \sim d_i$. Let $B > 0$ be an upper bound for the set $\{|b_j| : j \in J\} \cup \{|c_i| : i \in H\}$. Given $\epsilon > 0$, choose integers N and M such that

$$|a_i - c_i| < \frac{\epsilon}{2B} \tag{1.4.15}$$

for all $i > N$ and

$$|b_i - d_i| < \frac{\epsilon}{2B} \tag{1.4.16}$$

for all $i > M$. If L is the larger of N and M, then, for all $i > L$,

$$\begin{aligned} |a_i b_i - c_i d_i| &= |a_i b_i - b_i c_i + b_i c_i - c_i d_i| \\ &= |b_i (a_i - c_i) + c_i (b_i - d_i)| \\ &\leq |b_i (a_i - c_i)| + |c_i (b_i - d_i)| \\ &= |b_i| |a_i - c_i| + |c_i| |b_i - d_i| \\ &< B\frac{\epsilon}{2B} + B\frac{\epsilon}{2B} \\ &= \epsilon. \end{aligned} \tag{1.4.17}$$

Hence $a_i b_i \sim c_i d_i$. Thus if $u, v \in \mathbb{R}$, with u being the equivalence class of $\{a_i\}_{i \in I}$ and v being the equivalence class of $\{b_j\}_{j \in J}$, then we may unambiguously define uv to be the equivalence class of $\{a_i b_i\}_{i \in K}$, where $K = I \cap J$.

If $u \in \mathbb{R}$, we define $-u = (-1)u$. Note that if $\{a_i\}_{i \in I}$ is a Cauchy sequence of rational numbers in the equivalence class of u, then $\{-a_i\}_{i \in I}$ is a Cauchy sequence in the equivalence class of $-u$.

We will say that a sequence $\{a_i\}_{i \in I}$ is bounded away from 0 if there exists a rational number $\delta > 0$ and an integer N such that $|a_i| > \delta$ for all $i > N$. It should be clear that any sequence which converges to 0 is not bounded away from 0. Moreover, as a consequence of the next exercise, any Cauchy sequence which does not converge to 0 must be bounded away from 0.

? Exercise 1.4.2

Suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence which is not bounded away from 0. Show that the sequence converges and $\lim_{i\to\infty} a_i = 0$.

? Exercise 1.4.3

Suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence which is bounded away from 0 and $a_i \sim b_i$. Show that $\{b_j\}_{j\in J}$ is also bounded away from 0.

Now suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence which is bounded away from 0 and choose $\delta > 0$ and N so that $|a_i| > \delta$ for all $i > N$. Define a new sequence $\{c_i\}_{i=N+1}^{\infty} + 1$ by setting

$$c_i = \frac{1}{a_i}, i = N+1, N+2, ... \tag{1.4.18}$$

Given $\epsilon > 0$, choose M so that

$$|a_i - a_j| < \epsilon\delta^2 \tag{1.4.19}$$

for all $i, j > M$. Let L be the larger of N and M. Then, for all $i, j > L$, we have

$$\begin{aligned} |c_i - c_j| &= \left|\frac{1}{a_i} - \frac{1}{a_j}\right| \\ &= \left|\frac{a_j - a_i}{a_i a_j}\right| \\ &= \frac{|a_j - a_i|}{|a_i a_j|} \\ &< \frac{\epsilon\delta^2}{\delta^2} \\ &= \epsilon. \end{aligned} \tag{1.4.20}$$

Hence $\{c_i\}_{i=N+1}^{\infty}$ is a Cauchy sequence.

Now suppose $\{b_j\}_{j\in J}$ is a Cauchy sequence with $a_i \sim b_i$. By Exercise 1.4.3 we know that $\{b_j\}_{j\in J}$ is also bounded away from 0, so choose $\gamma > 0$ and K such that $|b_j| > \gamma$ for all $j > K$. Given $\epsilon > 0$, choose P so that

$$|a_i - b_i| < \epsilon\delta\gamma. \tag{1.4.21}$$

for all $i > P$. Let S be the larger of N, K, and P. Then, for all $i, j > S$, we have

$$\begin{aligned} \left|\frac{1}{a_i} - \frac{1}{b_i}\right| &= \left|\frac{b_i - a_i}{a_i b_i}\right| \\ &= \frac{|b_i - a_i|}{|a_i b_i|} \\ &< \frac{\epsilon\delta\gamma}{\delta\gamma} \\ &= \epsilon. \end{aligned} \tag{1.4.22}$$

Hence $\frac{1}{a_i} \sim \frac{1}{b_i}$. Thus if $u \neq 0$ is a real number which is the equivalence class of $\{a_i\}_{i\in I}$ (necessarily bounded away from 0), then we may define

$$a^{-1} = \frac{1}{a} \tag{1.4.23}$$

to be the equivalence class of

$$\left\{\frac{1}{a_i}\right\}_{i=N+1}^{\infty}, \tag{1.4.24}$$

where N has been chosen so that $|a_i| > \delta$ for all $i > N$ and some $\delta > 0$.

It follows immediately from these definitions that $\mathbb{R}$ is a field. That is:

1. $a + b = b + a$ for all $a, b \in \mathbb{R}$;

2. $(a+b)+c=a+(b+c)$ for all $a,b,c\in\mathbb{R}$;

3. $ab=ba$ for all $a,b\in\mathbb{R}$;

4. $(ab)c=a(bc)$ for all $a,b,c\in\mathbb{R}$;

5. $a(b+c)=ab+ac$ for all $a,b,c\in\mathbb{R}$;

6. $a+0=a$ for all $a\in\mathbb{R}$;

7. $a+(-a)=0$ for all $a\in\mathbb{R}$;

8. $1a=a$ for all $a\in\mathbb{R}$;

9. if $a\in\mathbb{R}, a\neq 0$, then $aa^{-1}=1$.

1.4.2 Order and Metric Properties

Definition

Given $u\in\mathbb{R}$, we say that u is positive, written $u>0$, if u is the equivalence class of a Cauchy sequence $\{a_i\}_{i\in I}$ for which there exists a rational number $\epsilon>0$ and an integer N such that $a_i>\epsilon$ for every $i>N$. A real number $u\in\mathbb{R}$ is said to be negative if $-u>0$. We let $\mathbb{R}^+$ denote the set of all positive real numbers.

Exercise 1.4.4

Show that if $u\in\mathbb{R}$, then one and only one of the following is true: $(\mathrm{a})u>0, (\mathrm{b})u<0$, or $(\mathrm{c})u=0$.

Exercise 1.4.5

Show that if $a,b\in\mathbb{R}^+$, then $a+b\in\mathbb{R}^+$.

Definition

Given real numbers u and v, we say u is greater than v, written $u>v$, or, equivalently, v is less than u, written, $v<u$, if $u-v>0$. We write $u\geq v$, or, equivalently, $v\leq u$, to indicate that u is either greater than or equal to v. We say that u is nonnegative if $u\geq 0$.

Exercise 1.4.6

Show that $\mathbb{R}$ is an ordered field, that is, verify the following:

a. For any $a,b\in\mathbb{R}$, one and only one of the following must hold: $(i)a<b$, (ii) $a=b, (\text{ iii})\ a>b$.

b. If $a,b,c\in\mathbb{R}$ with $a<b$ and $b<c$, then $a<c$.

c. If $a,b,c\in\mathbb{R}$ with $a<b$, then $a+c<b+c$.

d. If $a,b\in\mathbb{R}$ with $a>0$ and $b>0$, then $ab>0$.

Exercise 1.4.7

Show that if $a,b\in\mathbb{R}$ with $a>0$ and $b<0$, then $ab<0$.

Exercise 1.4.8

Show that if $a,b,c\in\mathbb{R}$ with $a<b$, then $ac<bc$ if $c>0$ and $ac>bc$ if $c<0$.

Exercise 1.4.9

Show that if $a, b \in \mathbb{R}$ with $a < b$, then for any real number λ with $0 < \lambda < 1, a < \lambda a + (1 - \lambda)b < b$.

Definition

For any $a \in \mathbb{R}$, we call

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0, \end{cases} \tag{1.4.25}$$

the absolute value of a.

Exercise 1.4.10

Show that for any $a \in \mathbb{R}, -|a| \leq a \leq |a|$.

Proposition 1.4.1

For any $a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$.

Proof

If $a + b \geq 0$, then

$$|a| + |b| - |a + b| = |a| + |b| - a - b = (|a| - a) + (|b| - b). \tag{1.4.26}$$

Both of the terms on the right are nonnegative by Exercise 1.4.10. Hence the sum is nonnegative and the proposition follows. If $a + b < 0$, then

$$|a| + |b| - |a + b| = |a| + |b| + a + b = (|a| + a) + (|b| + b). \tag{1.4.27}$$

Again, both of the terms on the right are nonnegative by Exercise 1.4.10. Hence the sum is nonnegative and the proposition follows. *Q.E.D.*

It is now easy to show that the absolute value function satisfies

1. $|a - b| \geq 0$ for all $a, b \in \mathbb{R}$, with $|a - b| = 0$ if and only if $a = b$,
2. $|a - b| = |b - a|$ for all $a, b \in \mathbb{R}$,
3. $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbb{R}$.

These properties show that the function

$$d(a, b) = |a - b| \tag{1.4.28}$$

is a metric, and we will call $|a - b|$ the distance from a to b.

Proposition 1.4.2

Given $a \in \mathbb{R}^+$, there exist $r, s \in \mathbb{Q}$ such that $0 < r < a < s$.

Proof

Let $\{u\}_{i \in I}$ be a Cauchy sequence in the equivalence class of a. Since $a > 0$, there exists a rational $\epsilon > 0$ and an integer N such that $u_i > \epsilon$ for all $i > N$. Let $r = \frac{\epsilon}{2}$. Then $u_i - r > \frac{\epsilon}{2}$ for every $i > N$, so $a - r > 0$, that is, $0 < r < a$.

Now choose an integer M so that $|u_i - u_j| < 1$ for all $i, j > M$. Let $s = u_{M+1} + 2$. Then

$$s - u_i = u_{M+1} + 2 - u_i > 1 \tag{1.4.29}$$

for all $i > M$. Hence $s > a$. Q.E.D.

Proposition 1.4.3

$\mathbb{R}$ is an archimedean ordered field.

Proof

Given real numbers a and b with $0 < a < b$, let r and s be rational numbers for which $0 < r < a < b < s$. since $\mathbb{Q}$ is a an archimedean field, there exists an integer n such that $nr > s$. Hence

$$na > nr > s > b. \tag{1.4.30}$$

Q.E.D.

Proposition 1.4.4

Given $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof

Let $\{u\}_{i\in I}$ be a Cauchy sequence in the equivalence class of a and let $\{v\}_{j\in J}$ be in the equivalence class of b. Since $b - a > 0$, there exists a rational $\epsilon > 0$ and an integer N such that $v_i - u_i > \epsilon$ for all $i > N$. Now choose an integer M so that $|u_i - u_j| < \frac{\epsilon}{4}$ for all $i, j > M$. Let $r = u_{M+1} + \frac{\epsilon}{2}$. Then

$$\begin{aligned} r - u_i &= u_{M+1} + \frac{\epsilon}{2} - u_i \\ &= \frac{\epsilon}{2} - (u_i - u_{M+1}) \\ &> \frac{\epsilon}{2} - \frac{\epsilon}{4} \\ &= \frac{\epsilon}{4} \end{aligned} \tag{1.4.31}$$

for all $i > M$ and

$$\begin{aligned} v_i - r &= v_i - u_{M+1} - \frac{\epsilon}{2} \\ &= (v_i - u_i) - (u_{M+1} - u_i) - \frac{\epsilon}{2} \\ &> \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{4} \end{aligned} \tag{1.4.32}$$

for all i larger than the larger of N and M. Hence $a < r < b$. Q.E.D.

1.4.3 Upper and Lower Bounds

Definition

Let $A \subset \mathbb{R}$. If $s \in \mathbb{R}$ is such that $s \geq a$ for every $a \in A$, then we call s an upper bound for A. If s is an upper bound for A with the property that $s \leq t$ whenever t is an upper bound for A, then we call s the supremum, or least upper bound, of A, denoted $s = \sup A$. Similarly, if $r \in \mathbb{R}$ is such that $r \leq a$ for every $a \in A$, then we call r a lower bound for A. If r is a lower bound for A with the property that $r \geq t$ whenever t is a lower bound for A, then we call r the infimum, or greatest lower bound, of A, denoted $r = \inf A$.

Theorem 1.4.5

Suppose $A \subset \mathbb{R}, A \neq \emptyset$, has an upper bound. Then sup A exists.

Proof

Let $a \in A$ and let b be an upper bound for A. Define sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows: Let $a_1 = a$ and $b_1 = b$. For $i > 1$, let

$$c = \frac{a_{i-1} + b_{i-1}}{2}. \tag{1.4.33}$$

If c is an upper bound for A, let $a_i = a_{i-1}$ and let $b_i = c_i$ otherwise, let $a_i = c$ and $b_i = b_{i-1}$. Then

$$|b_i - a_i| = \frac{|b-a|}{2^{i-1}} \tag{1.4.34}$$

for $i = 1, 2, 3, \ldots$ Now, for $i = 1, 2, 3, \ldots$, let r_i be a rational number such that $a_i < r_i < b_i$. Given any $\epsilon > 0$, we may choose N so that

$$2^N > \frac{|b-a|}{\epsilon}. \tag{1.4.35}$$

Then, whenever $i > N$ and $j > N$,

$$|r_i - r_j| < |b_{N+1} - a_{N+1}| = \frac{|b-a|}{2^N} < \epsilon. \tag{1.4.36}$$

Hence $\{r_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Let $s \in \mathbb{R}$ be the equivalence class of $\{r_i\}_{i=1}^{\infty}$. Note that, for $i = 1, 2, 3, \ldots, a_i \leq s \leq b_i$.

Now if s is not an upper bound for A, then there exists $a \in A$ with $a > s$. Let $\delta = a - s$ and choose an integer N such that

$$2^N > \frac{|b-a|}{\delta}. \tag{1.4.37}$$

Then

$$b_{N+1} \leq s + \frac{|b-a|}{2^N} < s + \delta = a. \tag{1.4.38}$$

But, by construction, b_{N+1} is an upper bound for A. Thus s must be an upper bound for A.

Now suppose t is another upper bound for A and $t < s$. Let $\delta = s - t$ and choose an integer N such that

$$2^N > \frac{|b-a|}{\delta}. \tag{1.4.39}$$

Then

$$a_{N+1} \geq s - \frac{|b-a|}{2^N} > s - \delta = t, \tag{1.4.40}$$

which implies that a_{N+1} is an upper bound for A. But, by construction, a_{N+1} is not an upper bound for A. Hence s must be the least upper bound for A, that is, $s = \sup A$. Q.E.D.

? Exercise 1.4.11

Show that if $A \subset \mathbb{R}$ is nonempty and has a lower bound, then inf A exists. (Hint: You may wish to first show that inf $A = -\sup(-A)$, where $-A = \{x : -x \in A\}$).

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CHAPTER OVERVIEW

2: Sequences and Series

2.1: Sequences

2.2: Infinite Series

Thumbnail: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above. (CC BY-SA-NC; OpenStax)

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2.1: Sequences

Definition

Let $\{a_i\}_{i\in I}$ be a sequence of real numbers. We say $\{a_i\}_{i\in I}$ converges, and has limit L, if for every real number $\epsilon > 0$ there exists an integer N such that

$$|a_i - L| < \epsilon \text{ whenever } i > N. \tag{2.1.1}$$

We say a sequence $\{a_i\}_{i\in I}$ which does not converge diverges.

Definitions: nondecreasing and nonincreasing sequences

- We say a sequence $\{a_i\}_{i\in I}$ is **nondecreasing** if $a_{i+1} \geq a_i$ for every $i \in I$ and increasing if $a_{i+1} > a_i$ for every $i \in I$.
- We say a sequence $\{a_i\}_{i\in I}$ is **nonincreasing** if $a_{i+1} \leq a_i$ for every $i \in I$ and decreasing if $a_{i+1} < a_i$ for every $i \in I$.

Definition: bounded sequences

We say a set $A \subset \mathbb{R}$ is bounded if there exists a real number M such that $|a| \leq M$ for every $a \in A$. We say a sequence $\{a_i\}_{i\in I}$ of real numbers is **bounded** if there exists a real number M such that $|a_i| \leq M$ for all $i \in I$.

Theorem 2.1.1

If $\{a_i\}_{i\in I}$ is a nondecreasing, bounded sequence of real numbers, then $\{a_i\}_{i\in I}$ converges.

Proof

Since $\{a_i\}_{i\in I}$ is bounded, the set of $A = \{a_i : i \in I\}$ has a supremum. Let $L = \sup A$. For any $\epsilon > 0$, there must exist $N \in I$ such that $a_N > L - \epsilon$ (or else $L - \epsilon$ would be an upper bound for A which is smaller than L). But then

$$L - \epsilon < a_N \leq a_i \leq L < L + \epsilon \tag{2.1.2}$$

for all $i \geq N$, that is,

$$|a_i - L| < \epsilon \tag{2.1.3}$$

for all $i \geq N$. Thus $\{a_i\}_{i\in I}$ converges and

$$L = \lim_{i\to\infty} a_i. \tag{2.1.4}$$

Q.E.D.

We conclude from the previous theorem that every nondecreasing sequence of real numbers either has a limit or is not bounded, that is, is unbounded.

Exercise 2.1.1

Show that a nonincreasing, bounded sequence of real numbers must converge.

Definition

Let $\{a_i\}_{i\in I}$ be a sequence of real numbers. If for every real number M there exists an integer N such that $a_i > M$ whenever $i > N$, then we say the sequence $\{a_i\}_{i\in I}$ diverges to positive infinity, denoted by

$$\lim_{i\to\infty} a_i = +\infty. \tag{2.1.5}$$

Similarly, if for every real number M there exists an integer N such that $a_i < M$ whenever $i > N$, then we say the sequence $\{a_i\}_{i\in I}$ diverges to negative infinity, denoted by

$$\lim_{i\to\infty} a_i = -\infty. \tag{2.1.6}$$

$$\lim_{i\to\infty} a_i = -\infty. \tag{2.1.6}$$

Exercise 2.1.2

Show that a nondecreasing sequence of real numbers either converges or diverges to positive infinity.

Exercise 2.1.3

Show that a nonincreasing sequence of real numbers either converges or diverges to negative infinity.

2.1.1 Extended Real Numbers

It is convenient to add the symbols $+\infty$ and $-\infty$ to the real numbers $\mathbb{R}$. Although neither $+\infty$ nor $-\infty$ is a real number, we agree to the following operational conventions:

1. Given any real number $r, -\infty < r < \infty$.

2. For any real number r,

$$r + (+\infty) = r - (-\infty) = r + \infty = +\infty, \tag{2.1.7}$$

$$r + (-\infty) = r - (+\infty) = r - \infty = -\infty, \tag{2.1.8}$$

and

$$\frac{r}{+\infty} = \frac{r}{-\infty} = 0. \tag{2.1.9}$$

3. For any real number $r > 0, r \cdot (+\infty) = +\infty$ and $r \cdot (-\infty) = -\infty$.

4. For any real number $r < 0, r \cdot (+\infty) = -\infty$ and $r \cdot (-\infty) = +\infty$.

5. If $a_i = -\infty, i = 1, 2, 3, \ldots$, then $\lim_{i\to\infty} a_i = -\infty$.

6. If $a_i = +\infty, i = 1, 2, 3, \ldots$, then $\lim_{i\to\infty} a_i = +\infty$.

Note that with the order relation defined in this manner, $+\infty$ is an upper

bound and $-\infty$ is a lower bound for any set $A \subset \mathbb{R}$. Thus if $A \subset \mathbb{R}$ does not have a finite upper bound, then $\sup A = +\infty$; similarly, if $A \subset \mathbb{R}$ does not have a finite lower bound, then $\inf A = -\infty$.

When working with extended real numbers, we refer to the elements of $\mathbb{R}$ as finite real numbers and the elements $+\infty$ and $-\infty$ as infinite real numbers.

Exercise 2.1.4

Do the extended real numbers form a field?

2.1.2 Limit Superior and Inferior

Definition

Let $\{a_i\}_{i\in I}$ be a sequence of real numbers and, for each $i \in I$, let $u_i = \sup\{a_j : j \geq i\}$. If $u_i = +\infty$ for every $i \in I$, we let

$$\limsup_{i\to\infty} a_i = +\infty; \tag{2.1.10}$$

otherwise, we let

$$\limsup_{i\to\infty} a_i = \inf\{u_i : i \in I\}. \tag{2.1.11}$$

In either case, we call $\liminf_{n\to\infty} a_n$ the limit inferior of the sequence $\{a_i\}_{i\in I}$.

Exercise 2.1.5

Given a sequence $\{a_i\}_{i\in I}$, define $\{u_i\}_{i\in I}$ and $\{l_i\}_{i\in I}$ as in the previous two definitions. Show that

$$\limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i \tag{2.1.12}$$

and

$$\liminf_{i\to\infty} a_i = \lim_{i\to\infty} l_i. \tag{2.1.13}$$

Exercise 2.1.6

Find $\limsup_{i\to\infty} a_i$ and $\liminf_{i\to\infty} a_i$ for the sequences $\{a_i\}_{i=1}^{\infty}$ as de fined below.

a. $a_i = (-1)^i$

b. $a_i = i$

c. $a_i = 2^{-i}$

d. $a_i = \frac{1}{i}$

The following proposition is often called the squeeze theorem.

Proposition 2.1.2

Suppose $\{a_i\}_{i\in I}, \{b_j\}_{j\in J}$, and $\{c_k\}_{k\in K}$ are sequences of real numbers for which there exists an integer N such that $a_i \le c_i \le b_i$ whenever $i > N$. If

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i, \tag{2.1.14}$$

then

$$\lim_{i\to\infty} c_i = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i. \tag{2.1.15}$$

Proof

Let $L = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$. Suppose L is finite. Given $\epsilon > 0$, there exists an integer M such that

$$|a_i - L| < \frac{\epsilon}{3} \tag{2.1.16}$$

and

$$|b_i - L| < \frac{\epsilon}{3} \tag{2.1.17}$$

whenever $i > M$. Then

$$|a_i - b_i| \le |a_i - L| + |L - b_i| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \tag{2.1.18}$$

whenever $i > M$. Let K be the larger of N and M. Then

$$|c_i - L| \le |c_i - b_i| + |b_i - L| \le |a_i - b_i| + |b_i - L| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \tag{2.1.19}$$

whenever $i > K$. Thus $\lim c_i = L$. The result when L is infinite is a consequence of the next two exercises. Q.E.D.

Exercise 2.1.7

Suppose $\{a_i\}_{i\in I}$ and $\{c_k\}_{k\in K}$ are sequences for which there exists an integer N such that $a_i \le c_i$ whenever $i > N$. Show that if $\lim_{i\to\infty} a_i = +\infty$, then $\lim_{i\to\infty} c_i = +\infty$.

Exercise 2.1.8

Suppose $\{b_j\}_{j\in J}$ and $\{c_k\}_{k\in K}$ are sequences for which there exists an integer N such that $c_i \leq b_i$ whenever $i > N$. Show that if $\lim_{i\to\infty} b_i = -\infty$, then $\lim_{i\to\infty} c_i = -\infty$.

Exercise 2.1.9

Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are sequences of real numbers with $a_i \leq b_i$ for all i larger than some integer N. Assuming both sequences converge, show that

$$\lim_{i\to\infty} a_i \leq \lim_{i\to\infty} b_i. \tag{2.1.20}$$

Exercise 2.1.10

Show that for any sequence $\{a_i\}_{i\in I}$,

$$\liminf_{i\to\infty} a_i \leq \limsup_{i\to\infty} n_i. \tag{2.1.21}$$

Proposition 2.1.3

Suppose $\{a_i\}_{i\in I}$ is a sequence for which

$$\limsup_{i\to\infty} a_i = \liminf_{i\to\infty} a_i. \tag{2.1.22}$$

Then

$$\lim_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = \liminf_{i\to\infty} a_i. \tag{2.1.23}$$

Proof

Let $u_i = \sup\{a_k : k \geq i\}$ and $l_i = \inf\{a_k : k \geq i\}$. Then $l_i \leq a_i \leq u_i$ for all $i \in I$. Now

$$\lim_{i\to\infty} l_i = \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i, \tag{2.1.24}$$

so the result follows from the squeeze theorem. Q.E.D.

Exercise 2.1.11

Suppose u is a real number such that $u \geq 0$ and $u < \epsilon$ for any real number $\epsilon > 0$. Show that $u = 0$.

2.1.3 Completeness

Definition

Suppose $\{a_i\}_{i\in I}$ is a sequence in $\mathbb{R}$. We call $\{a_i\}_{i\in I}$ a Cauchy sequence if for every $\epsilon > 0$ there exists an integer N such that

$$|a_i - a_j| < \epsilon \tag{2.1.25}$$

whenever both $i > N$ and $j > N$.

Theorem 2.1.4

Suppose $\{a_i\}_{i\in I}$ is a Cauchy sequence in $\mathbb{R}$. Then $\{a_i\}_{i\in I}$ converges to a limit $L \in \mathbb{R}$.

Proof

Let $u_i = \sup\{a_k : k \geq i\}$ and $l_i = \inf\{a_k : k \geq i\}$. Given any $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that $|a_i - a_j| < \epsilon$ for all $i, j > N$. Thus, for all $i, j > N$, $a_i < a_j + \epsilon_1$ and so

$$a_i \leq \inf\{a_j + \epsilon : j \geq i\} = l_i + \epsilon \tag{2.1.26}$$

for all $i > N$. Since $\{l_i\}_{i\in I}$ is a nondecreasing sequence,

$$a_i \leq \sup\{l_i + \epsilon : i \in I\} = \liminf_{i\to\infty} a_i + \epsilon \tag{2.1.27}$$

for all $i > N$. Hence

$$u_i = \sup\{a_k : k \geq i\} \leq \liminf_{i\to\infty} a_i + \epsilon \tag{2.1.28}$$

for all $i > N$. Thus

$$\limsup_{i\to\infty} a_i = \inf\{u_i : i \in I\} \leq \liminf_{i\to\infty} a_i + \epsilon. \tag{2.1.29}$$

Since $\liminf_{i\to\infty} a_i \leq \limsup_{i\to\infty} a_i$, it follows that

$$\left|\limsup_{i\to\infty} a_i - \liminf_{i\to\infty} a_i\right| \leq \epsilon. \tag{2.1.30}$$

since this is true for every $\epsilon > 0$, we have $\limsup_{i\to\infty} a_i = \lim_{i\to\infty} \inf a_i$, and so $\{a_i\}_{i\in I}$ converges by Proposition 2.1.3. Q.E.D.

As a consequence of the previous theorem, we say that $\mathbb{R}$ is a complete metric space.

? Exercise 2.1.12

Suppose $A \subset \mathbb{R}, A \neq \emptyset$, and $s = \sup A$. Show that there exists a sequence $\{a_i\}_{i=1}^{\infty}$ with $a_i \in A$ such that $\lim_{i\to\infty} a_i = s$.

? Exercise 2.1.13

Given a real number $x \geq 0$, show that there exists a real number $s \geq 0$ such that $s^2 = x$.

We let $\sqrt{x}$ denote the number s in the previous exercise, the square root of x.

2.1.4 Some Basic Results About Sequences

Proposition 2.1.5

Suppose $\{x_i\}_{i\in I}$ is a convergent sequence in $\mathbb{R}, \alpha$ is a real number, and $L = \lim_{i\to\infty} x_i$. Then the sequence $\{\alpha x_i\}_{i\in I}$ oonverges and

$$\lim_{i\to\infty} \alpha x_i = \alpha L. \tag{2.1.31}$$

Proof

If $\alpha = 0$, then $\{\alpha x_i\}_{i\in I}$ clearly converges to 0. So assume $\alpha \neq 0$. Given $\epsilon > 0$, choose an integer N such that

$$|x_i - L| < \frac{\epsilon}{|\alpha|} \tag{2.1.32}$$

whenever $i > N$. Then for any $i > N$ we have

$$|\alpha x_i - \alpha L| = |\alpha|\,|x_i - L| < |\alpha|\frac{\epsilon}{|\alpha|} = \epsilon. \tag{2.1.33}$$

Thus $\lim_{i\to\infty} \alpha x_i = \alpha L$. Q.E.D.

Proposition 2.1.6

Suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i\to\infty} x_i$ and $M = \lim_{i\to\infty} y_i$. Then the sequence $\{x_i + y_i\}_{i\in I}$ converges and

$$\lim_{i\to\infty} (x_i + y_i) = L + M. \tag{2.1.34}$$

Exercise 2.1.14

Prove the previous proposition.

Proposition 2.1.7

Suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i\to\infty} x_i$ and $M = \lim_{i\to\infty} y_i$. Then the sequence $\{x_i y_i\}_{i\in I}$ converges and

$$\lim_{i\to\infty} x_i y_i = LM. \tag{2.1.35}$$

Exercise 2.1.15

Prove the previous proposition.

Proposition 2.1.8

Suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i\to\infty} x_i$, $M = \lim_{i\to\infty} y_i$, and $y_i \neq 0$ for all $i \in I$. If $M \neq 0$, then the sequence $\left\{\frac{x_i}{y_i}\right\}_{i\in I}$ converges and

$$\lim_{i\to\infty} \frac{x_i}{y_i} = \frac{L}{M}. \tag{2.1.36}$$

Proof

Since $M \neq 0$ and $M = \lim_{i\to\infty} y_i$, we may choose an integer N such that

$$|y_i| > \frac{|M|}{2} \tag{2.1.37}$$

whenever $i > N$. Let B be an upper bound for $\{|x_i| : i \in I\} \cup \{|y_i| : i \in I\}$. Given any $\epsilon > 0$, we may choose an integer P such that

$$|x_i - L| < \frac{M^2\epsilon}{4B} \tag{2.1.38}$$

and

$$|y_i - M| < \frac{M^2\epsilon}{4B} \tag{2.1.39}$$

whenever $i > P$. Let K be the larger of N and P. Then, for any $i > K$, we have

$$\begin{aligned}
\left|\frac{x_i}{y_i} - \frac{L}{M}\right| &= \frac{|x_i M - y_i L|}{|y_i M|} \\
&= \frac{|x_i M - x_i y_i + x_i y_i - y_i L|}{|y_i M|} \\
&\leq \frac{|x_i|\,|M - y_i| + |y_i|\,|x_i - L|}{|y_i M|} \\
&< \frac{B\frac{M^2\epsilon}{4B} + B\frac{M^2\epsilon}{4B}}{\frac{M^2}{2}} \\
&= \epsilon.
\end{aligned} \tag{2.1.40}$$

Thus

$$\lim_{i\to\infty}\frac{x_i}{y_i}=\frac{L}{M}. \tag{2.1.41}$$

Q.E.D.

Exercise 2.1.16

a. Show that $\lim_{n\to\infty}\frac{1}{n}=0$.

b. Show that $\lim_{n\to\infty}\frac{1}{n^2}=0$ by (i) using the definition of limit directly and then (ii) using previous results.

Exercise 2.1.17

Show that for any positive integer k,

$$\lim_{n\to\infty}\frac{1}{n^k}=0. \tag{2.1.42}$$

Example 2.1.1

We may combine the properties of this section to compute

$$\begin{aligned}\lim_{n\to\infty}\frac{5n^3+3n-6}{2n^3+2n^2-7} &= \lim_{n\to\infty}\frac{5+\frac{3}{n^2}-\frac{6}{n^3}}{2+\frac{2}{n}-\frac{7}{n^3}}\\ &= \frac{\lim_{n\to\infty}5+3\lim_{n\to\infty}\frac{1}{n^2}-6\lim_{n\to\infty}\frac{1}{n^3}}{\lim_{n\to\infty}2+2\lim_{n\to\infty}\frac{1}{n}-7\lim_{n\to\infty}\frac{1}{n^3}}\\ &= \frac{5+0+0}{2+0+0}\\ &= \frac{5}{2}.\end{aligned} \tag{2.1.43}$$

Exercise 2.1.18

Evaluate

$$\lim_{n\to\infty}\frac{3n^5+8n^3-6n}{8n^5+2n^4-31}, \tag{2.1.44}$$

carefully showing each step.

Proposition 2.1.9

Suppose $\{x_i\}_{i\in I}$ is a convergent sequence of nonnegative real numbers with $L=\lim_{i\to\infty}x_i$. Then the sequence $\{\sqrt{x_i}\}_{i\in I}$ converges and

$$\lim_{i\to\infty}\sqrt{x_i}=\sqrt{L}. \tag{2.1.45}$$

Proof

Let $\epsilon>0$ be given. Suppose $L>0$ and note that

$$|x_i-L|=|\sqrt{x_i}-\sqrt{L}||\sqrt{x_i}+\sqrt{L}| \tag{2.1.46}$$

implies that

$$|\sqrt{x_i}-\sqrt{L}|=\frac{|x_i-L|}{|\sqrt{x_i}+\sqrt{L}|} \tag{2.1.47}$$

for any $i \in I$. Choose an integer N such that

$$|x_i - L| < \sqrt{L}\epsilon \tag{2.1.48}$$

whenever $i > N$. Then, for any $i > N$,

$$|\sqrt{x_i} - \sqrt{L}| = \frac{|x_i - L|}{|\sqrt{x_i} + \sqrt{L}|} < \frac{\sqrt{L}\epsilon}{\sqrt{L}} = \epsilon. \tag{2.1.49}$$

Hence $\lim_{i\to\infty} \sqrt{x_i} = \sqrt{L}$.

If $L = 0, \lim_{i\to\infty} x_i = 0$, so we may choose an integer N such that $|x_i| < \epsilon^2$ for all $i > N$. Then

$$|\sqrt{x_i}| < \epsilon \tag{2.1.50}$$

whenever $i > N$, so $\lim_{i\to\infty} \sqrt{x_i} = 0$. Q.E.D.

Exercise 2.1.19

Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+1}}{5n+6}, \tag{2.1.51}$$

carefully showing each step.

Exercise 2.1.20

Given real numbers $r > 0$ and α, show that $(a) \alpha r < r$ if $0 < \alpha < 1$ and $(b) r < \alpha r$ if $\alpha > 1$.

Proposition 2.1.10

If $x \in \mathbb{R}$ and $|x| < 1$, then

$$\lim_{n\to\infty} x^n = 0. \tag{2.1.52}$$

Proof

We first assume $x \geq 0$. Then the sequence $\{x^n\}_{n=1}^{\infty}$ is nonincreasing and bounded below by 0. Hence the sequence converges. Let $L = \lim_{n\to\infty} x^n$. Then

$$L = \lim_{n\to\infty} x^n = x \lim_{n\to\infty} x^{n-1} = xL, \tag{2.1.53}$$

from which it follows that $L(1-x) = 0$. since $1 - x > 0$, we must have $L = 0$. The result for $x < 0$ follows from the next exercise. Q.E.D.

Exercise 2.1.21

Show that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.

2.1.5 Subsequences

Definition

Given a sequence $\{x_i\}_{i=m}^{\infty}$, suppose $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of integers with

$$m \leq n_1 < n_2 < n_3 < \cdots. \tag{2.1.54}$$

Then we call the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ a subsequence of $\{x_i\}_{i=m}^{\infty}$.

Example 2.1.2

The sequence $\{x_{2k}\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{x_i\}_{i=1}^{\infty}$. For example, $\{\frac{1}{2i}\}_{i=1}^{\infty}$ is a subsequence of $\{\frac{1}{i}\}_{i=1}^{\infty}$.

Exercise 2.1.22

Suppose $\{x_i\}_{i=m}^{\infty}$ converges with $\lim_{i\to\infty} x_i = L$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also converges and $\lim_{k\to\infty} x_{n_k} = L$.

Exercise 2.1.23

Suppose $\{x_i\}_{i=m}^{\infty}$ diverges to $+\infty$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also diverges to $+\infty$.

Exercise 2.1.24

Suppose $\{x_i\}_{i=m}^{\infty}$ diverges to $-\infty$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also diverges to $-\infty$.

Definition

Given a sequence $\{x_i\}_{i=m}^{\infty}$, we call any extended real number λ which is the limit of a subsequence of $\{x_i\}_{i=m}^{\infty}$ a subsequential limit of $\{x_i\}_{i=m}^{\infty}$.

Example 2.1.3

-1 and 1 are both subsequential limits of $\{(-1)^i\}_{i=0}^{\infty}$.

Exercise 2.1.25

Suppose the sequence $\{x_i\}_{i=m}^{\infty}$ is not bounded. Show that either $-\infty$ or $+\infty$ is a subsequential limit of $\{x_i\}_{i=m}^{\infty}$..

Proposition 2.1.11

Suppose Λ is the set of all subsequential limits of the sequence $\{x_i\}_{i=m}^{\infty}$. Then $\Lambda \neq \emptyset$.

Proof

By the previous exercise, the proposition is true if $\{x_i\}_{i=m}^{\infty}$ is not bounded. So suppose $\{x_i\}_{i=m}^{\infty}$ is bounded and choose real numbers a and b such that $a \leq x_i \leq b$ for all $i \geq m$. Construct sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows: Let $a_1 = a$ and $b_1 = b$. For $i \geq 1$, let

$$c = \frac{a_{i-1} + b_{i-1}}{2}. \tag{2.1.55}$$

If there exists an integer N such that $a_{i-1} \leq x_j \leq c$ for all $j > N$, let $a_i = a_{i-1}$ and $b_i = c$; otherwise, let $a_i = c$ and $b_i = b_{i-1}$. Let $n_1 = m$ and, for $k = 2, 3, 4, \ldots$, let n_k be the smallest integer for which $n_k > n_{k-1}$ and $a_k \leq x_{n_k} \leq b_k$ Then $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence which is a subsequence of $\{x_i\}_{i=m}^{\infty}$. Thus $\{x_{n_k}\}_{k=1}^{\infty}$ converges and $\Lambda \neq 0$. Q.E.D.

Exercise 2.1.26

Suppose $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ with $a \leq b$ for every $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$.

Proposition 2.1.12

Let Λ be the set of subsequential limits of a sequence $\{x_i\}_{i=m}^{\infty}$. Then

$$\limsup_{i\to\infty} x_i = \sup \Lambda. \tag{2.1.56}$$

Proof

Let $s = \sup \Lambda$ and, for $i \geq m, u_i = \sup \{x_j : j \geq i\}$. Now since $x_j \leq u_i$ for all $j \geq i$, it follows that $\lambda \leq u_i$ for every $\lambda \in \Lambda$ and $i \geq m$. Hence, from the previous exercise, $s \leq \inf \{u_i : i \geq m\} = \limsup_{i \to \infty} x_i$.

Now suppose $s < \limsup_{i \to \infty} x_i$. Then there exists a real number t such that $s < t < \limsup_{i \to \infty} x_i$. In particular, $t < u_i$ for every $i \geq m$. Let n_1 be the smallest integer for which $n_1 \geq m$ and $x_{n_1} > t$. For $k = 2, 3, 4, \ldots$, let n_k be the smallest integer for which $n_k > n_{k-1}$ and $x_{n_k} > t$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_i\}_{i=m}^{\infty}$ which has a subsequential limit $\lambda \geq t$. Since λ is also then a subsequential limit of $\{x_i\}_{i=m}^{\infty}$, we have $\lambda \in \Lambda$ and $\lambda \geq t > s$, contradicting $s = \sup \Lambda$. Hence we must have $\limsup_{i \to \infty} x_i = \sup \Lambda$. Q.E.D.

? Exercise 2.1.27

Let Λ be the set of subsequential limits of a sequence $\{x_i\}_{i=m}^{\infty}$. Show that

$$\liminf_{i \to \infty} x_i = \inf \Lambda. \tag{2.1.57}$$

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2.2: Infinite Series

Definition

Given a sequence $\{a_i\}_{i=m}^{\infty}$, let $\{s_n\}_{n=m}^{\infty}$ be the sequence defined by

$$s_n = \sum_{i=m}^{n} a_i. \tag{2.2.1}$$

We call the sequence $\{s_n\}_{n=m}^{\infty}$ an infinite series. If $\{s_n\}_{n=m}^{\infty}$ converges, we call

$$s = \lim_{n\to\infty} s_n \tag{2.2.2}$$

the sum of the series. For any integer $n \geq m$, we call s_n a partial sum of the series.

We will use the notation

$$\sum_{i=1}^{\infty} a_i \tag{2.2.3}$$

to denote either $\{s_n\}_{n=m}^{\infty}$, the infinite series, or s, the sum of the infinite series. Of course, if $\{s_n\}_{n=m}^{\infty}$ diverges, then we say $\sum_{i=m}^{\infty} a_i$ diverges.

Exercise 2.2.1

Suppose $\sum_{i=m}^{\infty} a_i$ converges and $\beta \in \mathbb{R}$. Show that $\sum_{i=m}^{\infty} \beta a_i$ also converges and

$$\sum_{i=m}^{\infty} \beta a_i = \beta \sum_{i=m}^{\infty} a_i. \tag{2.2.4}$$

Exercise 2.2.2

Suppose both $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ converge. Show that $\sum_{i=m}^{\infty} (a_i + b_i)$ converges and

$$\sum_{i=m}^{\infty} (a_i + b_i) = \sum_{i=m}^{\infty} a_i + \sum_{i=m}^{\infty} b_i. \tag{2.2.5}$$

Exercise 2.2.3

Given an infinite series $\sum_{i=m}^{\infty} a_i$ and an integer $k \geq m$, show that $\sum_{i=m}^{\infty} a_i$ converges if and only if $\sum_{i=k}^{\infty} a_i$ converges.

Proposition 2.2.1

Suppose $\sum_{i=m}^{\infty} a_i$ converges. Then $\lim_{n\to\infty} a_n = 0$.

Proof

Let $s_n = \sum_{i=m}^{n} a_i$ and $s = \lim_{n\to\infty} s_n$. Since $a_n = s_n - s_{n-1}$, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$. Q.E.D.

Exercise 2.2.4

Let $s = \sum_{i=0}^{\infty} (-1)^n$. Note that

$$s = \sum_{n=0}^{\infty} (-1)^n = 1 - \sum_{n=0}^{\infty} (-1)^n = 1 - s, \tag{2.2.6}$$

from which it follows that $s = \frac{1}{2}$. Is this correct?

Exercise 2.2.5

Show that for any real number $x \neq 1$,

$$s_n = \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}. \tag{2.2.7}$$

(Hint: Note that $x^{n+1} = s_{n+1} - s_n = 1 + xs_n - s_n$.)

Theorem 2.2.2

For any real number x with $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}. \tag{2.2.8}$$

Proof

If $s_n = \sum_{i=0}^{n} x^i$, then, by the previous exercise,

$$s_n = \frac{1 - x^{n+1}}{1 - x}. \tag{2.2.9}$$

Hence

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}. \tag{2.2.10}$$

2.2.1 Comparison Tests

The following two propositions are together referred to as the comparison test.

Proposition 2.2.3

Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N such that $0 \leq a_i \leq b_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} b_i$ converges, then $\sum_{i=m}^{\infty} a_i$ converges.

Proof

By Exercise 2.2 .3 We need only show that $\sum_{i=N}^{\infty} a_i$ converges. Let s_n be the nth partial sum of $\sum_{i=N}^{\infty} a_i$ and let t_n be the nth partial sum of $\sum_{i=N}^{\infty} b_i$. Now

$$s_{n+1} - s_n = a_{n+1} \geq 0 \tag{2.2.11}$$

for every $n \geq N$, so $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing sequence. Moreover,

$$s_n \leq t_n \leq \sum_{i=N}^{\infty} b_i < +\infty \tag{2.2.12}$$

for every $n \geq N$. Thus $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing, bounded sequence, and so converges.

Proposition 2.2.4

Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N such that $0 \leq a_i \leq b_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} a_i$ diverges then $\sum_{i=m}^{\infty} b_i$ diverges.

Proof

By Exercise 2.2 .3 we need only show that $\sum_{i=N}^{\infty} b_i$ diverges. Let s_n be the nth partial sum of $\sum_{i=N}^{\infty} a_i$ and let t_n be the nth partial sum of $\sum_{i=N}^{\infty} b_i$. Now $\{s_n\}_{n=N}^{\infty}, N$ is a nondecreasing sequence which diverges, and so we must have $\lim_{n\to\infty} s_n$

$= +\infty$. Thus given any real number M there exists an integer L such that

$$M < s_n \leq t_n \tag{2.2.13}$$

whenever $n > L$. Hence $\lim_{n\to\infty} t_n = +\infty$ and $\sum_{i=m}^{\infty} b_i$ diverges. (\quad\) Q.E.D.

Example 2.2.1

Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots. \tag{2.2.14}$$

Now for $n = 1, 2, 3, \ldots$, we have

$$0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}}. \tag{2.2.15}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \tag{2.2.16}$$

converges, it follows that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \tag{2.2.17}$$

converges. Moreover,

$$2 < \sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{1 - \frac{1}{2}} = 3. \tag{2.2.18}$$

We let

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{2.2.19}$$

Proposition 2.2.5

$e \notin \mathbb{Q}$.

Proof

Suppose $e = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$. Let

$$a = q! \left(e - \sum_{i=0}^{q} \frac{1}{n!} \right). \tag{2.2.20}$$

Then $a \in \mathbb{Z}^+$ since $q!e = (q-1)!p$ and $n!$ divides $q!$ when $n \leq q$. At the same time

$$\begin{aligned} a &= q! \left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{i=0}^{q} \frac{1}{n!} \right) \\ &= q! \sum_{n=q+1}^{\infty} \frac{1}{n!} \\ &= \left(\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \right) \\ &= \frac{1}{q+1} \left(1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots \right) \\ &= \frac{1}{q+1} \left(1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots \right) \end{aligned} \tag{2.2.21}$$

$$
\begin{aligned}
&< \frac{1}{q+1}\left(1+\frac{1}{q+1}+\frac{1}{(q+1)^2}+\cdots\right)\\
&= \frac{1}{q+1}\sum_{n=0}^{\infty}\frac{1}{(q+1)^n}\\
&= \frac{1}{q+1}\left(\frac{1}{1-\frac{1}{q+1}}\right)\\
&= \frac{1}{q}
\end{aligned}
$$

Since this is impossible, we conclude that no such integers p and q exist. Q.E.D.

Definition

We call a real number which is not a rational number an irrational number.

Example 2.2.2

We have seen that $\sqrt{2}$ and e are irrational numbers.

Proposition 2.2.6

Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N and a real number $M > 0$ such that $0 \leq a_i \leq Mb_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} b_i$ converges, then $\sum_{i=m}^{\infty} a_i$ converges.

Proof

Since $\sum_{i=k}^{\infty} Mb_i$ converges whenever $\sum_{i=k}^{\infty} b_i$ does, the result follows from the comparison test. Q.E.D.

Exercise 2.2.6

Suppose $\sum_{i=m}^{\infty} a_i$ diverges. Show that $\sum_{i=m}^{\infty} \beta a_i$ diverges for any real number $\beta \neq 0$.

Proposition 2.2.7

Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer N and a real number $M > 0$ such that $0 \leq a_i \leq Mb_i$ whenever $i \geq N$. If $\sum_{i=m}^{\infty} a_i$ diverges, then $\sum_{i=k}^{\infty} b_i$ diverges.

Proof

By the comparison test, $\sum_{i=m}^{\infty} Mb_i$ diverges. Hence, by the previous exercise, $\sum_{i=m}^{\infty} b_i$ also diverges.

We call the results of the next two exercises, which are direct consequences of the last two propositions, the limit comparison test.

Exercise 2.2.7

Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} b_i$ converges and

$$\lim_{i\to\infty} \frac{a_i}{b_i} < +\infty, \tag{2.2.22}$$

then $\sum_{i=m}^{\infty} a_i$ converges.

Exercise 2.2.8

Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} a_i$ diverges and

$$\lim_{i\to\infty} \frac{a_i}{b_i} < +\infty, \tag{2.2.23}$$

then $\sum_{i=m}^{\infty} b_i$ diverges.

Exercise 2.2.9

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} \tag{2.2.24}$$

converges.

Exercise 2.2.10

Show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2.2.25}$$

converges for any real number $x \geq 0$.

Exercise 2.2.11

Let S be the set of all finite sums of numbers in the set $\{a_1, a_2, a_3, ...\}$, where $a_i \geq 0$ for $i = 1, 2, 3, ...$ That is

$$S = \left\{ \sum_{i\in J} a_i : J \subset \{1, 2, 3, ..., n\} \text{ for some } n \in \mathbb{Z}^+ \right\}. \tag{2.2.26}$$

Show that $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sup S < \infty$, in which case

$$\sum_{i=1}^{\infty} a_i = \sup S. \tag{2.2.27}$$

One consequence of the preceding exercise is that the sum of a sequence of nonnegative numbers depends only on the numbers begin added, and not on the order in which they are added. That is, if $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$ is one-to-one and onto, $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sum_{i=1}^{\infty} a_{\varphi(i)}$ converges, and, in that case,

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\varphi(i)}. \tag{2.2.28}$$

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CHAPTER OVERVIEW

3: Cardinality

3.1: Binary Representations

3.2: Countable and Uncountable Sets

3.3: Power Sets

Thumbnail: Bijective function from N to the set E of even numbers. Although E is a proper subset of N, both sets have the same cardinality. (CC BY-SA 3.0; Jochen Burghardt via Wikipedia)

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3.1: Binary Representations

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that, for each $n = 1, 2, 3, \ldots$, either $a_n = 0$ or $a_n = 1$ and, for any integer N, there exists an integer $n > N$ such that $a_n = 0$. Then

$$0 \leq \frac{a_n}{2^n} \leq \frac{1}{2^n} \tag{3.1.1}$$

for $n = 1, 2, 3, \ldots$, so the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n} \tag{3.1.2}$$

converges to some real number x by the comparison test. Moreover,

$$0 \leq x < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \tag{3.1.3}$$

We call the sequence $\{a_n\}_{n=1}^{\infty}$ the binary representation for x, and write

$$x = .a_1 a_2 a_3 a_4 \ldots. \tag{3.1.4}$$

? Exercise 3.1.1

Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both binary representations for x. Show that $a_n = b_n$ for $n = 1, 2, 3, \ldots$.

Now suppose $x \in \mathbb{R}$ with $0 \leq x < 1$. Construct a sequence $\{a_n\}_{n=1}^{\infty}$ as follows: If $0 \leq x < \frac{1}{2}$, let $a_1 = 0$; otherwise, let $a_1 = 1$. For $n = 1, 2, 3, \ldots$, let

$$s_n = \sum_{i=1}^{n} \frac{a_i}{2^i} \tag{3.1.5}$$

and set $a_{n+1} = 1$ if

$$s_n + \frac{1}{2^{n+1}} \leq x \tag{3.1.6}$$

and $a_{n+1} = 0$ otherwise.

lemma 3.1.1

With the notation as above,

$$s_n \leq x < s_n + \frac{1}{2^n} \tag{3.1.7}$$

for $n = 1, 2, 3, \ldots$.

Proof

Since

$$s_1 = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq x < 1 \end{cases} \tag{3.1.8}$$

it is clear that $s_1 \leq x < s_1 + \frac{1}{2}$. So suppose $n > 1$ and $s_{n-1} \leq x < s_{n-1} + \frac{1}{2^{n-1}}$. If $s_{n-1} + \frac{1}{2^n} \leq x$, then $a_n = 1$ and

$$s_n = s_{n-1} + \frac{1}{2^n} \leq x < s_{n-1} + \frac{1}{2^{n-1}} = s_{n-1} + \frac{1}{2^n} + \frac{1}{2^n} = s_n + \frac{1}{2^n}. \tag{3.1.9}$$

If $x < s_{n-1} + \frac{1}{2^n}$, then $a_n = 0$ and

$$s_n = s_{n-1} \leq x < s_{n-1} + \frac{1}{2^n} = s_n + \frac{1}{2^n}. \tag{3.1.10}$$

Q.E.D.

Proposition 3.1.2

With the notation as above,

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}. \tag{3.1.11}$$

Proof

Given $\epsilon > 0$, choose an integer N such that $\frac{1}{2^n} < \epsilon$. Then, for any $n > N$, it follows from the lemma that

$$|s_n - x| < \frac{1}{2^n} < \frac{1}{2^N} < \epsilon. \tag{3.1.12}$$

Hence

$$x = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \frac{a_n}{2^n}. \tag{3.1.13}$$

Q.E.D.

lemma 3.1.3

With the notation as above, given any integer N there exists an integer $n > N$ such that $a_n = 0$.

Proof

If $a_n = 1$ for $n = 1, 2, 3, \ldots$, then

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \tag{3.1.14}$$

contradicting the assumption that $0 \leq x < 1$. Now suppose there exists an integer N such that $a_N = 0$ but $a_n = 1$ for every $n > N$. Then

$$x = s_N + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = s_{N-1} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = s_{N-1} + \frac{1}{2^N}, \tag{3.1.15}$$

implying that $a_N = 1$, and thus contradicting the assumption that $a_N = 0$. Q.E.D.

Combining the previous lemma with the previous proposition yields the following result.

Proposition 3.1.4

With the notation as above, $x = .a_1a_2a_3a_4\ldots$.

The next theorem now follows from Exercise 3.1.1 and the previous proposition.

Theorem 3.1.5

Every real number $0 \leq x < 1$ has a unique binary representation.

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3.2: Countable and Uncountable Sets

Definition

A function $\varphi : A \to B$ is said to be a one-to-one correspondence if φ is both one-to-one and onto.

Definition

We say sets A and B have the same cardinality if there exists a one-to-one correspondence $\varphi : A \to B$.

We denote the fact A and B have the same cardinality by writing $|A| = |B|$.

Exercise 3.2.1

Define a relation on sets by setting $A \sim B$ if and only if $|A| = |B|$. Show that this relation is an equivalence relation.

Definition

Let A be a set. If, for $n \in \mathbb{Z}^+$, A has the cardinality of the set $\{1, 2, 3, \ldots, n\}$, we say A is finite and write $|A| = n$. If A has the cardinality of $\mathbb{Z}^+$, we say A is countable and write $|A| = \aleph_0$.

Example 3.2.1

If we define $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ by

$$\varphi(n) = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd,} \\ -\frac{n}{2}, & \text{if } n \text{ is even,} \end{cases} \tag{3.2.1}$$

then φ is a one-to-one correspondence. Thus $|\mathbb{Z}| = \aleph_0$.

Exercise 3.2.2

Let A be the set of even integers. Show that $|A| = \aleph_0$.

Exercise 3.2.3

Verify each of the following:

a. If A is a nonempty subset of $\mathbb{Z}^+$, then A is either finite or countable.

b. If A is a nonempty subset of a countable set B, then A is either finite or countable.

Proposition 3.2.1

Suppose A and B are countable sets. Then the set $C = A \cup B$ is countable.

Proof

Suppose A and B are disjoint, that is, $A \cap B = \emptyset$. Let $\varphi : \mathbb{Z}^+ \to A$ and $\psi : \mathbb{Z}^+ \to B$ be one-to-one correspondences. Define $\tau : \mathbb{Z}^+ \to C$ by

$$\tau(n) = \begin{cases} \varphi\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd,} \\ \psi\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases} \tag{3.2.2}$$

Then τ is a one-to-one correspondence, showing that C is countable.

If A and B are not disjoint, then τ is onto but not one-to-one. However, in that case C has the cardinality of an infinite subset of $\mathbb{Z}^+$, and so is countable. Q.E.D.

Definition

A nonempty set which is not finite is said to be infinite. An infinite set which is not countable is said to be uncountable.

Exercise 3.2.4

Suppose A is uncountable and $B \subset A$ is countable. Show that $A \backslash B$ is uncountable.

Proposition 3.2.2

Suppose A and B are countable. Then $C = A \times B$ is countable.

Proof

Let $\varphi : \mathbb{Z}^+ \to A$ and $\psi : \mathbb{Z}^+ \to B$ be one-to-one correspondences. Let $a_i = \varphi(i)$ and $b_i = \psi(i)$. Define $\tau : \mathbb{Z}^+ \to C$ by letting

$$\tau(1) = (a_1, b_1)\,, \tag{3.2.3}$$

$$\tau(2) = (a_1, b_2)\,, \tag{3.2.4}$$

$$\tau(3) = (a_2, b_1)\,, \tag{3.2.5}$$

$$\tau(4) = (a_1, b_3)\,, \tag{3.2.6}$$

$$\tau(5) = (a_2, b_2)\,, \tag{3.2.7}$$

$$\tau(6) = (a_3, b_1)\,, \tag{3.2.8}$$

$$\tau(7) = (a_1, b_4)\,, \tag{3.2.9}$$

$$\vdots = \vdots \tag{3.2.10}$$

That is, form the infinite matrix with (a_i, b_j) in the ith row and jth column, and then count the entries by reading down the diagonals from right to left. Then τ is a one-to-one correspondence and C is countable.

Proposition 3.2.3

$\mathbb{Q}$ is countable.

Proof

By the previous proposition, $\mathbb{Z} \times \mathbb{Z}$ is countable. Let

$$A = \{(p, q) : p, q \in \mathbb{Z}, q > 0, p \text{ and } q \text{ relatively prime }.\} \tag{3.2.11}$$

Then A is infinite and $A \subset \mathbb{Z} \times \mathbb{Z}$, so A is countable. But clearly $|\mathbb{Q}| = |A|$, so $\mathbb{Q}$ is countable. Q.E.D.

Proposition 3.2.4

Suppose for each $i \in \mathbb{Z}^+$, A_i is countable. Then

$$B = \bigcup_{i=1}^{\infty} A_i \tag{3.2.12}$$

is countable.

Proof

Suppose the sets $A_i, i \in \mathbb{Z}^+$, are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ for all $i, j \in \mathbb{Z}^+$. For each $i \in \mathbb{Z}^+$, let $\varphi_i : \mathbb{Z}^+ \to A_i$ be a one-to-one correspondence. Then $\psi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to B$ defined by

$$\psi(i, j) = \varphi_i(j) \tag{3.2.13}$$

is a one-to-one correspondence, and so $|B| = |\mathbb{Z}^+ \times \mathbb{Z}^+| = \aleph_0$.

If the sets $A_i, i \in \mathbb{Z}^+$, are not disjoint, then ψ is onto but not one-to-one. But then there exists a subset P of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that $\psi : P \to B$ is a one-to-one correspondence. Since P is an infinite subset of a countable set, P is countable and so $|B| = \aleph_0$. Q.E.D.

If in the previous proposition we allow that, for each $i \in \mathbb{Z}^+$, A_i is either finite or countable, then $B = \bigcup_{i=1}^{\infty} A_i$ will be either finite or countable.

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3.3: Power Sets

Definition

Given a set A, we call the set of all subsets of A the power set of A, which we denote $\mathcal{P}(A)$.

Example 3.3.1

If $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}. \tag{3.3.1}$$

Proposition 3.3.1

If A is finite with $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof

Let

$$B = \{(b_1, b_2, \ldots, b_n) : b_i = 0 \text{ or } b_i = 1, i = 1, 2, \ldots, n\} \tag{3.3.2}$$

and let $a_1, a_2, \ldots, a_n$ be the elements of A. Define $\varphi : B \to \mathcal{P}(A)$ by letting

$$\varphi(b_1, b_2, \ldots, b_n) = \{a_i : b_i = 1, i = 1, 2, \ldots, n\}. \tag{3.3.3}$$

Then φ is a one-to-one correspondence. The result now follows from the next exercise. Q.E.D.

Exercise 3.3.1

With B as in the previous proposition, show that $|B| = 2^n$.

In analogy with the case when A is finite, we let $2^{|A|} = |\mathcal{P}(A)|$ for any nonempty set A.

Definition

Suppose A and B are sets for which there exists a one-to-one function $\varphi : A \to \bar{B}$ but there does not exist a one-to-one correspondence $\psi : A \to B$. Then we write $|A| < |B|$.

Theorem 3.3.2

If A is a nonempty set, then $|A| < |\mathcal{P}(A)|$.

Proof

Define $\varphi : A \to \mathcal{P}(\mathbb{Z}^+)$ by

$$\varphi(\{a_i\}_{i=1}^{\infty}) = \{i : i \in \mathbb{Z}^+, a_i = 1\}. \tag{3.3.4}$$

Then φ is a one-to-one correspondence. Q.E.D.

Now let B be the set of all sequences $\{a_i\}_{i=1}^{\infty}$ in A such that for every integer N there exists an integer $n > N$ such that $a_n = 0$. Let $C = A \backslash B$,

$$D_0 = \{\{a_i\}_{i=1}^{\infty} : a_i = 1, i = 1, 2, 3, \ldots\}, \tag{3.3.5}$$

and

$$D_j = \{\{a_i\}_{i=1}^{\infty} : a_j = 0, a_k = 1 \text{ for } k > j\} \tag{3.3.6}$$

for $j = 1, 2, 3, \ldots$ Then $|D_0| = 1$ and $|D_j| = 2^{j-1}$ for $j = 1, 2, 3, \ldots$ Moreover,

$$C = \bigcup_{j=0}^{\infty} D_j, \tag{3.3.7}$$

so C is countable. Since $A = B \cup C$, and A is uncountable, it follows that B is uncountable. Now if we let

$$I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}, \tag{3.3.8}$$

we have seen that the function $\varphi : B \to I$ defined by

$$\varphi\left(\{a_i\}_{i=1}^{\infty}\right) = a_1 a_2 a_3 a_4 \ldots \tag{3.3.9}$$

is a one-to-one correspondence. It follows that I is uncountable. As a consequence, we have the following result.

Theorem 3.3.4

$\mathbb{R}$ is uncountable.

Exercise 3.3.2

Let $I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}$. Show that

a. $|I| = |\{x : x \in \mathbb{R}, 0 \leq x \leq 1\}|$

b. $|I| = |\{x : x \in \mathbb{R}, 0 < x < 1\}|$

c. $|I| = |\{x : x \in \mathbb{R}, 0 < x < 2\}|$

d. $|I| = |\{x : x \in \mathbb{R}, -1 < x < 1\}|$

Exercise 3.3.3

Let $I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}$ and suppose a and b are real numbers with $a < b$. Show that

$$|I| = |\{x : x \in \mathbb{R}, a \leq x < b\}|. \tag{3.3.10}$$

Exercise 3.3.4

Does there exist a set $A \subset \mathbb{R}$ for which $\aleph_0 < |A| < 2^{\aleph_0}$? (Before working too long on this problem, you may wish to read about Cantor's continum hypothesis.)

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CHAPTER OVERVIEW

4: Topology of the Real Line

4.1: Intervals
4.2: Open Sets
4.3: Closed Sets
4.4: Compact Sets

Thumbnail: The blue circle represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$. The red disk represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$. The red set is an open set, the blue set is its boundary set, and the union of the red and blue sets is a closed set. (Public Domain; Richard Giuly via Wikipedia)

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4.1: Intervals

Definition: intervals

Given any two extended real numbers $a < b$, we call the set

$$(a, b) = \{x : x \in \mathbb{R}, a < x < b\} \tag{4.1.1}$$

an **open interval**.

Given any two finite real numbers $a \leq b$, we call the sets

$$[a, b] = \{x : x \in \mathbb{R}, a \leq x \leq b\}, \tag{4.1.2}$$

$$(-\infty, b] = \{x : x \in \mathbb{R}, x \leq b\}, \tag{4.1.3}$$

and

$$[a, +\infty) = \{x : x \in \mathbb{R}, x \geq a\} \tag{4.1.4}$$

closed intervals.

We call any set which is an open interval, a closed interval, or is given by, for some finite real numbers $a < b$,

$$(a, b] = \{x : x \in \mathbb{R}, a < x \leq b\} \tag{4.1.5}$$

or

$$[a, b) = \{x : x \in \mathbb{R}, a \leq x < b\}, \tag{4.1.6}$$

an **interval**.

Proposition 4.1.1

If $a, b \in \mathbb{R}$ with $a < b$, then

$$(a, b) = \{x : x = \lambda a + (1 - \lambda)b, 0 < \lambda < 1\}. \tag{4.1.7}$$

Proof

Suppose $x = \lambda a + (1 - \lambda)b$ for some $0 < \lambda < 1$. Then

$$b - x = \lambda b - \lambda a = \lambda(b - a) > 0, \tag{4.1.8}$$

so $x < b$. Similarly,

$$x - a = (\lambda - 1)a + (1 - \lambda)b = (1 - \lambda)(b - a) > 0, \tag{4.1.9}$$

so $a < x$. Hence $x \in (a, b)$.

Now suppose $x \in (a, b)$. Then

$$x = \left(\frac{b - x}{b - a}\right) a + \left(\frac{x - a}{b - a}\right) b = \left(\frac{b - x}{b - a}\right) a + \left(1 - \frac{b - x}{b - a}\right) b \tag{4.1.10}$$

and

$$0 < \frac{b - x}{b - a} < 1. \tag{4.1.11}$$

Q.E.D.

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4.2: Open Sets

Definition

We say a set $U \subset \mathbb{R}$ is open if for every $x \in U$ there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U. \tag{4.2.1}$$

Proposition 4.2.1

Every open interval I is an open set.

Proof

Suppose $I = (a, b)$, where $a < b$ are extended real numbers. Given $x \in I$, let ϵ be the smaller of $x - a$ and $b - x$. Suppose $y \in (x - \epsilon, x + \epsilon)$. If $b = +\infty$, then $b > y$; otherwise, we have

$$b - y > b - (x + \epsilon) = (b - x) - \epsilon \geq (b - x) - (b - x) = 0, \tag{4.2.2}$$

so $b > y$. If $a = -\infty$, then $a < y$; otherwise,

$$y - a > (x - \epsilon) - a = (x - a) - \epsilon \geq (x - a) - (x - a) = 0, \tag{4.2.3}$$

so $a < y$. Thus $y \in I$ and I is an open set. Q.E.D.

Note that $\mathbb{R}$ is an open set (it is, in fact, the open interval $(-\infty, +\infty)$), as is $\emptyset$ (it satisfies the definition trivially).

Proposition 4.2.2

Suppose A is a set and, for each $\alpha \in A, U_\alpha$ is an open set. Then

$$\bigcup_{\alpha \in A} U_\alpha \tag{4.2.4}$$

is an open set.

Proof

Let $x \in \cup_{\alpha \in A} U_\alpha$. Then $x \in U_\alpha$ for some $\alpha \in A$. Since U_α is open, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_\alpha$. Thus

$$(x - \epsilon, x + \epsilon) \subset U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha. \tag{4.2.5}$$

Hence $\bigcup_{\alpha \in A} U_\alpha$ is open. Q.E.D.

Proposition 4.2.3

Suppose $U_1, U_2, \ldots, U_n$ is a finite collection of open sets. Then

$$\bigcap_{i=1}^{n} U_i \tag{4.2.6}$$

is open.

Proof

Let $x \in \bigcap_{i=1}^{n} U_i$. Then $x \in U_i$ for every $i = 1, 2, \ldots, n$. For each i, choose $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Let ϵ be the smallest of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. Then $\epsilon > 0$ and

$$(x - \epsilon, x + \epsilon) \subset (x - \epsilon_i, x + \epsilon_i) \subset U_i \tag{4.2.7}$$

for every $i = 1, 2, \ldots, n$. Thus

$$(x-\epsilon, x+\epsilon) \subset \bigcap_{i=1}^{n} U_i. \tag{4.2.8}$$

Hence $\bigcap_{i=1}^{n} U_i$ is an open set. Q.E.D.

Definition

Let $A \subset \mathbb{R}$. We say $x \in A$ is an interior point of A if there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset A$. We call the set of all interior points of A the interior of A, denoted A°.

Exercise 4.2.1

Show that if $A \subset \mathbb{R}$, then A° is open.

Exercise 4.2.2

Show that A is open if and only if $A = A^\circ$.

Exercise 4.2.3

Let $U \subset \mathbb{R}$ be a nonempty open set. Show that $\sup U \notin U$ and $\inf U \notin U$.

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4.3: Closed Sets

Definition: limit points

We call a point $x \in \mathbb{R}$ a **limit point** of a set $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $a \in A, a \neq x$, such that $a \in (x-\epsilon, x+\epsilon)$.

Definition: isolated point

Suppose $A \subset \mathbb{R}$. We call a point $a \in A$ an **isolated point** of A if there exists an $\epsilon > 0$ such that

$$A \cap (a-\epsilon, a+\epsilon) = \{a\}. \tag{4.3.1}$$

Exercise 4.3.1

Identify the limit points and isolated points of the following sets:

a. $[-1,1]$,

b. $(-1,1)$,

c. $\left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$,

d. $\mathbb{Z}$,

e. $\mathbb{Q}$.

Exercise 4.3.2

Suppose x is a limit point of the set A. Show that for every $\epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap A$ is infinite.

We let A' denote the set of limit points of a set A.

Definition: closures

Given a set $A \subset \mathbb{R}$, we call the set $\bar{A} = A \cup A'$ the **closure** of A.

Definition: Closed sets

We call a set $C \subset \mathbb{R}$ **closed** if $C = \bar{C}$.

Proposition 4.3.1

If $A \subset \mathbb{R}$, then $\bar{A}$ is closed.

Proof

Suppose x is a limit point of $\bar{A}$. We we will show that x is a limit point of A, and hence $x \in \bar{A}$. Now for any $\epsilon > 0$, there exists $a \in \bar{A}, a \neq x$, such that

$$a \in \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right). \tag{4.3.2}$$

If $a \in A$, let $b = a$. If $a \notin A$, then a is a limit point of A, so there exists $b \in A$, $b \neq a$ and $b \neq x$, such that

$$b \in \left(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right). \tag{4.3.3}$$

In either case

$$|x-b| \leq |x-a| + |a-b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.3.4}$$

Hence $x \in A'$, and so $\bar{A}$ is closed. Q.E.D.

Proposition 4.3.2

A set $C \subset \mathbb{R}$ is closed if and only if for every convergent sequence $\{a_k\}_{k\in K}$ with $a_k \in C$ for all $k \in K$,

$$\lim_{k\to\infty} a_k \in C. \tag{4.3.5}$$

Proof

Suppose C is closed and $\{a_k\}_{k\in K}$ is a convergent sequence with $a_k \in C$ for all $k \in K$. Let $x = \lim_{k\to\infty} a_k$. If $x = a_k$ for some integer k, then $x \in C$. Otherwise, for every $\epsilon > 0$, there exists an integer N such that $|a_N - x| < \epsilon$. Hence $a_N \neq x$ and

$$a_N \in (x - \epsilon, x + \epsilon). \tag{4.3.6}$$

Thus x is a limit point of C, and so $x \in C$ since C is closed.

Now suppose that for every convergent sequence $\{a_k\}_{k\in K}$ with $a_k \in C$ for all $k \in K, \lim_{k\to\infty} a_k \in C$. Let x be a limit point of C. For $k = 1, 2, 3, \ldots$, choose $a_k \in C$ such that $a_k \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$. Then clearly

$$x = \lim_{k\to\infty} a_k, \tag{4.3.7}$$

so $x \in C$. Thus C is closed. Q.E.D.

Exercise 4.3.3

Show that every closed interval I is a closed set.

Proposition 4.3.3

Suppose A is a set and, for each $\alpha \in A, C_\alpha$ is a closed set. Then

$$\bigcap_{\alpha\in A} C_\alpha \tag{4.3.8}$$

is a closed set.

Proof

Suppose x is a limit point of $\bigcap_{\alpha\in A} C_\alpha$. Then for any $\epsilon > 0$, there exists $y \in \bigcap_{\alpha\in A} C_\alpha$ such that $y \neq x$ and $y \in (x - \epsilon, x + \epsilon)$. But then for any $\alpha \in A$, $y \in C_\alpha$, so x is a limit point of C_α. Since C_α is closed, it follows that $x \in C_\alpha$ for every $\alpha \in A$. Thus $x \in \bigcap_{\alpha\in A} C_\alpha$ and $\bigcap_{\alpha\in A} C_\alpha$ is closed. Q.E.D.

Proposition 4.3.4

Suppose $C_1, C_2, \ldots, C_n$ is a finite collection of closed sets. Then

$$\bigcup_{i=1}^{n} C_i \tag{4.3.9}$$

is closed.

Proof

Suppose $\{a_k\}_{k\in K}$ is a convergent sequence with $a_k \in \bigcup_{i=1}^{n} C_i$ for every $k \in K$. Let $L = \lim_{k\to\infty} a_k$. Since K is an infinite set, there must an integer m and a subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ such that $a_{n_j} \in C_m$ for $j = 1, 2, \ldots$. Since every subsequence of $\{a_k\}_{k\in K}$ converges to $L, \{a_{n_j}\}_{j=1}^{\infty}$ must converge to L. Since C_m is closed,

$$L = \lim_{j\to\infty} a_{n_j} \in C_m \subset \bigcup_{i=1}^{n} C_i. \tag{4.3.10}$$

Thus $\bigcup_{i=1}^{n} C_i$ is closed. Q.E.D.

Note that both $\mathbb{R}$ and $\emptyset$ satisfy the definition of a closed set.

Proposition 4.3.5

A set $C \subset \mathbb{R}$ is closed if and only if $\mathbb{R} \backslash C$ is open.

Proof

Assume C is closed and let $U = \mathbb{R} \backslash C$. If $C = \mathbb{R}$, then $U = \emptyset$, which is open; if $C = \emptyset$, then $U = \mathbb{R}$, which is open. So we may assume both C and U are nonempty. Let $x \in U$. Then x is not a limit point of C, so there exists an $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \cap C = \emptyset. \tag{4.3.11}$$

Thus

$$(x - \epsilon, x + \epsilon) \subset U, \tag{4.3.12}$$

so U is open.

Now suppose $U = \mathbb{R} \backslash C$ is open. If $U = \mathbb{R}$, then $C = \emptyset$, which is closed; if $U = \emptyset$, then $C = \mathbb{R}$, which is closed. So we may assume both U and C are nonempty. Let x be a limit point of C. Then, for every $\epsilon > 0$,

$$(x - \epsilon, x + \epsilon) \cap C \neq \emptyset. \tag{4.3.13}$$

Hence there does not exist $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U. \tag{4.3.14}$$

Thus $x \notin U$, so $x \in C$ and C is closed. Q.E.D.

Exercise 4.3.4

For $n = 1, 2, 3, \ldots$, let $I_n = \left(-\frac{1}{n}, \frac{n+1}{n}\right)$. Is

$$\bigcap_{n=1}^{\infty} I_n \tag{4.3.15}$$

open or closed?

Exercise 4.3.5

For $n = 3, 4, 5, \ldots$, let $I_n = \left[\frac{1}{n}, \frac{n-1}{n}\right]$. Is

$$\bigcup_{n=3}^{\infty} I_n \tag{4.3.16}$$

open or closed?

Exercise 4.3.6

Suppose, for $n = 1, 2, 3, \ldots$, the intervals $I_n = [a_n, b_n]$ are such that $I_{n+1} \subset I_n$. If $a = \sup\{a_n : n \in \mathbb{Z}^+\}$ and $b = \inf\{b_n : n \in \mathbb{Z}^+\}$, show that

$$\bigcap_{n=1}^{\infty} I_n = [a, b]. \tag{4.3.17}$$

Exercise 4.3.7

Find a sequence $I_n, n = 1, 2, 3, \ldots$, of closed intervals such that $I_{n+1} \subset I_n$ for $n = 1, 2, 3, \ldots$ and

$$\bigcap_{n=1}^{\infty} I_n = \emptyset. \tag{4.3.18}$$

Exercise 4.3.8

Find a sequence $I_n, n = 1, 2, 3, \ldots$, of bounded, open intervals such that $I_{n+1} \subset I_n$ for $n = 1, 2, 3, \ldots$ and

$$\bigcap_{n=1}^{\infty} I_n = \emptyset. \tag{4.3.19}$$

Exercise 4.3.9

Suppose $A_i \subset \mathbb{R}, i = 1, 2, \ldots, n$, and let $B = \bigcup_{i=1}^{n} A_i$. Show that

$$\overline{B} = \bigcup_{i=1}^{n} \overline{A_i}. \tag{4.3.20}$$

Exercise 4.3.10

Suppose $A_i \subset \mathbb{R}, i \in \mathbb{Z}^+$, and let

$$B = \bigcup_{i=1}^{\infty} A_i. \tag{4.3.21}$$

Show that

$$\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}. \tag{4.3.22}$$

Find an example for which

$$\overline{B} \neq \bigcup_{i=1}^{\infty} \overline{A_i}. \tag{4.3.23}$$

Exercise 4.3.11

Suppose $U \subset \mathbb{R}$ is a nonempty open set. For each $x \in U$, let

$$J_x = \bigcup (x - \epsilon, x + \delta), \tag{4.3.24}$$

where the union is taken over all $\epsilon > 0$ and $\delta > 0$ such that $(x - \epsilon, x + \delta) \subset U$.

a. Show that for every $x, y \in U$, either $J_x \cap J_y = \emptyset$ or $J_x = J_y$.

b. Show that

$$U = \bigcup_{x \in B} J_x, \tag{4.3.25}$$

where $B \subset U$ is either finite or countable.

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4.4: Compact Sets

Definition

Suppose $T \subset \mathbb{R}$. If A is a set, U_α is an open set for every $\alpha \in A$, and

$$T \subset \bigcup_{\alpha \in A} U_\alpha, \tag{4.4.1}$$

then we call $\{U_\alpha : \alpha \in A\}$ an open cover of T.

4.4.1 Example

For $n = 3, 4, 5, \ldots$, let

$$U_n = \left(\frac{1}{n}, \frac{n-1}{n}\right). \tag{4.4.2}$$

Then $\{U_n : n = 3, 4, 5, \ldots\}$ is an open cover of the open interval $(0, 1)$.

Definition

Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of $T \subset \mathbb{R}$. If $B \subset A$ and

$$T \subset \bigcup_{\beta \in B} U_\beta, \tag{4.4.3}$$

then we call $\{U_\beta : \beta \in B\}$ a subcover of $\{U_\alpha : \alpha \in A\}$. If B is finite, we call $\{U_\beta : \beta \in B\}$ a finite subcover of $\{U_\alpha : \alpha \in A\}$.

4.4.1? Exercise

Show that the open cover of $(0, 1)$ given in the previous example does not have a finite subcover.

Definition

We say a set $K \subset \mathbb{R}$ is compact if every open cover of K has a finite sub cover.

4.4.2 Example

As a consequence of the previous exercise, the open interval $(0, 1)$ is not compact.

4.4.2? Exercise

Show that every finite subset of $\mathbb{R}$ is compact.

4.4.3? Exercise

Suppose $n \in \mathbb{Z}^+$ and $K_1, K_2, \ldots, K_n$ are compact sets. Show that $\bigcup_{i=1}^{n} K_i$ is compact.

4.4.1 proposition

If I is a closed, bounded interval, then I is compact.

Proof

Let $a \leq b$ be finite real numbers and $I = [a, b]$. Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of I. Let $\mathcal{O}$ be the set of sets $\{U_\beta : \beta \in B\}$ with the properties that B is a finite subset of A and $a \in \bigcup_{\beta \in B} U_\beta$. Let

$$(s - \epsilon, s + \epsilon) \subset U_\alpha. \tag{4.4.4}$$

Moreover, there exists a $\{U_\beta : \beta \in B\} \in \mathcal{O}$ for which

$$\left[a, s - \frac{\epsilon}{2}\right] \subset \bigcup_{\beta \in B} U_\beta. \tag{4.4.5}$$

But then

$$\{U_\beta : \beta \in B\} \cup \{U_\alpha\} \in \mathcal{O} \tag{4.4.6}$$

and

$$\left[a, s + \frac{\epsilon}{2}\right] \subset \left(\bigcup_{\beta \in B} U_\beta\right) \cup U_\alpha, \tag{4.4.7}$$

contradicting the definition of s. Hence we must have $s = b$. Now choose U_α such that $b \in U_\alpha$. Then, for some $\epsilon > 0$,

$$(b - \epsilon, b + \epsilon) \subset U_\alpha. \tag{4.4.8}$$

Moreover, there exists $\{U_\beta : \beta \in B\} \in \mathcal{O}$ such that

$$\left[a, b - \frac{\epsilon}{2}\right] \subset \bigcup_{\beta \in B} U_\beta. \tag{4.4.9}$$

Then

$$\{U_\beta : \beta \in B\} \cup \{U_\alpha\} \in \mathcal{O} \tag{4.4.10}$$

is a finite subcover of I. Thus I is compact. Q.E.D.

4.4.2 proposition

If K is a closed, bounded subset of $\mathbb{R}$, then K is compact.

Proof

Since K is bounded, there exist finite real numbers a and b such that $K \subset [a, b]$. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of K. Let $V = \mathbb{R} \backslash K$. Then

$$\{U_\alpha : \alpha \in A\} \cup \{V\} \tag{4.4.11}$$

in the latter case, we have

$$K \subset [a, b] \backslash V \subset \bigcup_{\beta \in B} U_\beta. \tag{4.4.12}$$

In either case, we have found a finite subcover of $\{U_\alpha : \alpha \in A\}$. Q.E.D.

4.4.4? Exercise

Show that if K is compact and $C \subset K$ is closed, then C is compact.

4.4.3 proposition

If $K \subset \mathbb{R}$ is compact, then K is closed.

Proof

Suppose x is a limit point of K and $x \notin K$. For $n = 1, 2, 3, \ldots$, let

$$U_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right). \tag{4.4.13}$$

Then

$$\bigcup_{n=1}^{\infty} U_n = (-\infty, x) \cup (x, +\infty) \supset K \tag{4.4.14}$$

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$$\bigcup_{n=1}^{\infty} U_n = (-\infty, x) \cup (x, +\infty) \supset K. \tag{4.4.14}$$

However, for any $N \in \mathbb{Z}^+$, there exists $a \in K$ with

$$a \in \left(x - \frac{1}{N}, x + \frac{1}{N}\right), \tag{4.4.15}$$

and hence

$$a \notin \bigcup_{n=1}^{N} U_n = \left(-\infty, x - \frac{1}{N}\right) \cup \left(x + \frac{1}{N}, +\infty\right). \tag{4.4.16}$$

Thus the open cover $\{U_n : n \in \mathbb{Z}^+\}$ does not have a finite subcover, contradicting the assumption that K is compact. Q.E.D.

4.4.5? Exercise

Suppose that for each α in some set A, K_α is compact. Show that $\bigcap_{\alpha \in A} K_\alpha$ is compact.

4.4.4 proposition

If $K \subset \mathbb{R}$ is compact, then K is bounded.

Proof

Suppose K is not bounded. For $n = 1, 2, 3, \ldots$, let $U_n = (-n, n)$. Then

$$\bigcup_{n=1}^{\infty} U_n = (-\infty, \infty) \supset K. \tag{4.4.17}$$

But, for any integer N, there exists $a \in K$ such that $|a| > N$, from which it follows that

$$a \notin \bigcup_{n=1}^{N} U_n = (-N, N). \tag{4.4.18}$$

Thus the open cover $\{U_n : n \in \mathbb{Z}^+\}$ does not have a finite subcover, contradicting the assumption that K is compact. Q.E.D.

Taken together, the previous three propositions yield the following fundamental result:

4.4.5 Theorem

A set $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.

4.4.6 proposition

If $K \subset \mathbb{R}$ is compact and $\{x_n\}_{n \in I}$ is a sequence with $x_n \in K$ for every $n \in I$, then $\{x_n\}_{n \in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} x_{n_k} \in K. \tag{4.4.19}$$

Proof

Since K is bounded, $\{x_n\}_{n \in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Since K is closed, we must have $\lim_{k \to \infty} x_{n_k} \in K$.

4.4.7 proposition

Suppose $K \subset \mathbb{R}$ is such that whenever $\{x_n\}_{n\in I}$ is a sequence with $x_n \in K$ for every $n \in I$, then $\{x_n\}_{n\in I}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} \in K$. Then K is compact.

Proof

Suppose K is unbounded. Then we may construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in K$ and $|x_n| > n$ for $n = 1, 2, 3, ...$ Hence the only possible subsequential limits of $\{x_n\}_{n=1}^{\infty}$ would be $-\infty$ and $+\infty$, contradicting our assumptions. Thus K must be bounded.

Now suppose $\{x_n\}_{n\in I}$ is a convergent sequence with $x_n \in K$ for all $n \in I$. If $L = \lim_{n\to\infty} x_n$, then L is the only subsequential limit of $\{x_n\}_{n\in I}$. Hence, by the assumptions of the proposition, $L \in K$. Hence K is closed.

Since K is both closed and bounded, it is compact. Q.E.D.

4.4.6? Exercise

Show that a set $K \subset \mathbb{R}$ is compact if and only if every infinite subset of K has a limit point in K.

4.4.7? Exercise

Show that if K is compact, then $\sup K \in K$ and $\inf K \in K$.

4.4.8 Theorem

Given a set $K \subset \mathbb{R}$, the following are equivalent:

1. Every open cover of K has a finite subcover.
2. Every sequence in K has a subsequential limit in K.
3. Every infinite subset of K has a limit point in K.

4.4.8? Exercise

Suppose $K_1, K_2, K_3, ...$ are nonempty compact sets with

$$K_{n+1} \subset K_n \tag{4.4.20}$$

for $n = 1, 2, 3, ...$ Show that

$$\bigcap_{n=1}^{\infty} K_n \tag{4.4.21}$$

is nonempty.

4.4.9? Exercise

We say a collection of sets $\{D_\alpha : \alpha \in A\}$ has the finite intersection property if for every finite set $B \subset A$,

$$\bigcap_{\alpha\in B} D_\alpha \neq \emptyset. \tag{4.4.22}$$

Show that a set $K \subset \mathbb{R}$ is compact if and only for any collection

$$\{E_\alpha : \alpha \in A, E_\alpha = C_\alpha \cap K \text{ where } C_\alpha \subset \mathbb{R} \text{ is closed } \} \tag{4.4.23}$$

which has the finite intersection property we have

$$\bigcap_{\alpha\in A} E_\alpha \neq \emptyset. \tag{4.4.24}$$

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CHAPTER OVERVIEW

5: Limits and Continuity

5.1: Limits
5.2: Monotonic Functions
5.3: Limits to Infinity and Infinite Limits
5.4: Continuous Functions

Thumbnail: The function $f(x) = 1/(x-a)^n$ has infinite limits at a. (CC BY; OpenStax)

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5.1: Limits

Let $A \subset \mathbb{R}$ and let x be a limit point of A. In the following, we will let $S(A,x)$ denote the set of all convergent sequences $\{x_n\}_{n\in I}$ such that $x_n \in A$ for all $n \in I, x_n \neq x$ for all $n \in I$, and $\lim_{n\to\infty} x_n = x$. We will let $S^+(A,x)$ be the subset of $S(A,x)$ of sequences $\{x_n\}_{n\in I}$ for which $x_n > x$ for all $n \in I$ and $S^-(A,x)$ be the subset of $S(A,x)$ of sequences $\{x_n\}_{n\in I}$ for which $x_n < x$ for all $n \in I$.

Definition

Let $D \subset \mathbb{R}, f : D \to \mathbb{R}, L \in \mathbb{R}$, and suppose a is a limit point of D. We say the limit of f as x approaches a is L, denoted

$$\lim_{x\to a} f(x) = L, \tag{5.1.1}$$

if for every sequence $\{x_n\}_{n\in I} \in S(D,a)$,

$$\lim_{n\to\infty} f(x_n) = L. \tag{5.1.2}$$

If $S^+(D,a) \neq \emptyset$, we say the limit from the right of f as x approaches a is L, denoted

$$\lim_{x\to a+} f(x) = L, \tag{5.1.3}$$

if for every sequence $\{x_n\}_{n\in I} \in S^+(D,a)$,

$$\lim_{n\to\infty} f(x_n) = L, \tag{5.1.4}$$

and, if $S^-(D,a) \neq \emptyset$, we say the limit from the left of f as x approaches a is L, denoted

$$\lim_{x\to a-} f(x) = L, \tag{5.1.5}$$

if for every sequence $\{x_n\}_{n\in I} \in S^-(D,a)$,

$$\lim_{n\to\infty} f(x_n) = L. \tag{5.1.6}$$

We may also denote

$$\lim_{x\to a} f(x) = L \tag{5.1.7}$$

by writing

$$f(x) \to L \text{ as } x \to a. \tag{5.1.8}$$

Similarly, we may denote

$$\lim_{x\to a^+} f(x) = L \tag{5.1.9}$$

by writing

$$f(x) \to L \text{ as } x \downarrow a \tag{5.1.10}$$

and

$$\lim_{x\to a^-} f(x) = L \tag{5.1.11}$$

by writing

$$f(x) \to L \text{ as } x \uparrow a \tag{5.1.12}$$

We also let

$$f(a+) = \lim_{x\to a^+} f(x) \tag{5.1.13}$$

and

$$f(a-) = \lim_{x\to a^-} f(x). \tag{5.1.14}$$

It should be clear that if $\lim_{x\to a} f(x) = L$ and $S^+(D,a) \neq \emptyset$, then $f(a+) = L$. Similarly, if $\lim_{x\to a} f(x) = L$ and $S^-(D,a) \neq \emptyset$, then $f(a-) = L$.

Proposition 5.1.1

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is a limit point of D. If $f(a-) = f(a+) = L$, then $\lim_{x\to a} f(x) = L$.

Proof

Suppose $\{x_n\}_{n=m}^{\infty} \in S(D,a)$. Let

$$J^- = \{n : n \in \mathbb{Z}, x_n < a\} \tag{5.1.15}$$

and

$$J^+ = \{n : n \in \mathbb{Z}, x_n > a\}. \tag{5.1.16}$$

Suppose J^- is empty or finite and let $k = m - 1$ if $J^- = \emptyset$ and, otherwise, let k be the largest integer in J^-. Then $\{x_n\}_{n=k+1}^{\infty} \in S^+(D,a)$, and so

$$\lim_{n\to\infty} f(x_n) = f(a+) = L. \tag{5.1.17}$$

A similar argument shows that if J^+ is empty or finite, then

$$\lim_{n\to\infty} f(x_n) = f(a-) = L. \tag{5.1.18}$$

If neither J^- nor J^+ is finite or empty, then $\{x_n\}_{n\in J^-}$ and $\{x_n\}_{n\in J^+}$ are subsequences of $\{x_n\}_{n=m}^{\infty}$ with $\{x_n\}_{n\in J^-} \in S^-(D,a)$ and $\{x_n\}_{n\in J^+} \in S^+(D,a)$. Hence, given any $\epsilon > 0$, we may find integers N and M such that

$$|f(x_n) - L| < \epsilon \tag{5.1.19}$$

whenever $n \in \{j : j \in J^-, j > N\}$ and

$$|f(x_n) - L| < \epsilon \tag{5.1.20}$$

whenever $n \in \{j : j \in J^+, j > M\}$. Let P be the larger of N and M. Since $J^- \cup J^+ = \{j : j \in \mathbb{Z}^+, j \geq m\}$, it follows that

$$|f(x_n) - L| < \epsilon \tag{5.1.21}$$

whenever $n > P$. Hence $\lim_{n\to\infty} f(x_n) = L$, and so $\lim_{x\to a} f(x) = L$. Q.E.D.

Proposition 5.1.2

Suppose $D \subset \mathbb{R}$, a is a limit point of D, and $f : D \to \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\alpha \in \mathbb{R}$, then

$$\lim_{x\to a} \alpha f(x) = \alpha L. \tag{5.1.22}$$

Proof

Suppose $\{x_n\}_{n\in I} \in S(D,a)$. Then

$$\lim_{n\to\infty} \alpha f(x_n) = \alpha \lim_{n\to\infty} f(x_n) = \alpha L. \tag{5.1.23}$$

Hence $\lim_{x\to a} \alpha f(x) = \alpha L$. Q.E.D.

Proposition 5.1.3

Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}(f(x) + g(x)) = L + M. \tag{5.1.24}$$

Proof

Suppose $\{x_n\}_{n\in I} \in S(D,a)$. Then

$$\lim_{n\to\infty}(f(x_n)+g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L+M. \tag{5.1.25}$$

Hence $\lim_{x\to a}(f(x)+g(x)) = L+M$. Q.E.D.

Proposition 5.1.4

Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f: D \to \mathbb{R}$, and $g: D \to \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} f(x)g(x) = LM. \tag{5.1.26}$$

Exercise 5.1.1

Prove the previous proposition.

Proposition 5.1.5

Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$, and $g(x) \neq 0$ for all $x \in D$. If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}. \tag{5.1.27}$$

Exercise 5.1.2

Prove the previous proposition.

Proposition 5.1.6

Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f: D \to \mathbb{R}$, and $f(x) \geq 0$ for all $x \in D$. If $\lim_{x\to a} f(x) = L$, then

$$\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}. \tag{5.1.28}$$

Exercise 5.1.3

Prove the previous proposition.

Given $D \subset \mathbb{R}$, $f: D \to \mathbb{R}$, and $A \subset D$, we let

$$f(A) = \{y : y = f(x) \text{ for some } x \in A\}. \tag{5.1.29}$$

In particular, $f(D)$ denotes the range of f.

Proposition 5.1.7

Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, a is a limit point of D, $g: D \to \mathbb{R}$, $f: E \to \mathbb{R}$, and $g(D) \subset E$. Moreover, suppose $\lim_{x\to a} g(x) = b$ and, for some $\epsilon > 0$, $g(x) \neq b$ for all $x \in (a-\epsilon, a+\epsilon) \cap D$. If $\lim_{x\to b} f(x) = L$, then

$$\lim_{x\to a} f \circ g(x) = L. \tag{5.1.30}$$

Proof

Suppose $\{x_n\}_{n\in I} \in S(D,a)$. Then

$$\lim_{n\to\infty} g(x_n) = b. \tag{5.1.31}$$

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Let $N \in \mathbb{Z}^+$ such that $|x_n - a| < \epsilon$ whenever $n > N$. Then

$$\{g(x_n)\}_{n=N+1}^{\infty} \in S(E, b), \tag{5.1.32}$$

so

$$\lim_{n\to\infty} f(g(x_n)) = L. \tag{5.1.33}$$

Thus $\lim_{x\to a} f \circ g(x) = L$. Q.E.D.

Example 5.1.1

Let

$$g(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{5.1.34}$$

If $f(x) = g(x)$, then

$$f \circ g(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{5.1.35}$$

Hence $\lim_{x\to 0} f \circ g(x) = 1$, although $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} f(x) = 0$.

5.1.1 Limits of Polynomials and Rational Functions

Example 5.1.2

If $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = c$ for all $x \in \mathbb{R}$, then clearly $\lim_{x\to a} f(x) = c$ for any $a \in \mathbb{R}$.

Example 5.1.3

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in \mathbb{R}$. If, for any $a \in \mathbb{R}$, $\{x_n\}_{n\in I} \in S(\mathbb{R}, a)$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = a. \tag{5.1.36}$$

Hence $\lim_{x\to a} x = a$.

Example 5.1.4

Suppose $n \in \mathbb{Z}^+$ and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^n$. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^n = \prod_{i=1}^{n} \lim_{x\to a} x = a^n. \tag{5.1.37}$$

Definition

If $n \in \mathbb{Z}, n \geq 0$, and $b_0, b_1, \ldots, b_n$ are real numbers with $b_n \neq 0$, then we call the function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \tag{5.1.38}$$

a polynomial of degree n.

Exercise 5.1.4

Show that if f is a polynomial and $a \in \mathbb{R}$, then $\lim_{x\to a} f(x) = f(a)$.

Definition

Suppose p and q are polynomials and

$$D = \{x : x \in \mathbb{R}, q(x) \neq 0\}. \tag{5.1.39}$$

We call the function $r : D \to \mathbb{R}$ defined by

$$r(x) = \frac{p(x)}{q(x)} \tag{5.1.40}$$

a rational function.

Exercise 5.1.5

Show that if f is a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

Exercise 5.1.6

Suppose $D \subset \mathbb{R}, a \in D$ is a limit point of D, and $\lim_{x\to a} f(x) = L$. If $E = D\backslash\{a\}$ and $g : E \to \mathbb{R}$ is defined by $g(x) = f(x)$ for all $x \in E$, show that $\lim_{x\to a} g(x) = L$.

Exercise 5.1.7

Evaluate

$$\lim_{x\to 1} \frac{x^5 - 1}{x^3 - 1}. \tag{5.1.41}$$

Exercise 5.1.8

Suppose $D \subset \mathbb{R}, a$ is a limit point of $D, f : D \to \mathbb{R}, g : D \to \mathbb{R}, h : D \to \mathbb{R}$, and $f(x) \leq h(x) \leq g(x)$ for all $x \in D$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L$, show that $\lim_{x\to a} h(x) = L$. (This is the squeeze theorem for limits of functions.)

Note that the above results which have been stated for limits will hold as well for the appropriate one-sided limits, that is, limits from the right or from the left.

Exercise 5.1.9

Suppose

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 4, & \text{if } x = 0, \\ x^2, & \text{if } x > 0. \end{cases} \tag{5.1.42}$$

Evaluate $f(0), f(0-)$, and $f(0+)$. Does $\lim_{x\to 0} f(x)$ exist?

5.1.2 Equivalent Definitions

Proposition 5.1.8

Suppose $D \subset \mathbb{R}, a$ is a limit point of D, and $f : D \to \mathbb{R}$. Then $\lim_{x\to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \neq a \text{ and } x \in (a - \delta, a + \delta) \cap D. \tag{5.1.43}$$

Proof

Suppose $\lim_{x\to a} f(x) = L$. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in (a-\delta, a+\delta) \cap D, x \neq a$, for which $|f(x) - L| \geq \epsilon$. For $n = 1, 2, 3, \ldots$, choose

$$x_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \cap D, \tag{5.1.44}$$

$x_n \neq a$, such that $|f(x_n) - L| \geq \epsilon$. Then $\{x_n\}_{n=1}^{\infty} \in S(D, a)$, but $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L, contradicting the assumption that $\lim_{x\to a} f(x) = L$.

Now suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \neq a$ and $x \in (a-\delta, a+\delta) \cap D$. Let $\{x_n\}_{n\in I} \in S(D, a)$. Given $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - L| < \epsilon$ whenever $x \neq a$ and $x \in (a-\delta, a+\delta) \cap D$. Choose $N \in \mathbb{Z}$ such that $|x_n - a| < \delta$ whenever $n > N$. Then $|f(x_n) - L| < \epsilon$ for all $n > N$. Hence $\lim_{n\to\infty} f(x_n) = L$, and so $\lim_{x\to a} f(x) = L$. Q.E.D.

The proofs of the next two propositions are analogous.

Proposition 5.1.9

Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $S^-(D, a) \neq \emptyset$. Then $\lim_{x\to a^-} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a-\delta, a) \cap D. \tag{5.1.45}$$

Proposition 5.1.10

Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $S^+(D, a) \neq \emptyset$. Then $\lim_{x\to a^+} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a, a+\delta) \cap D. \tag{5.1.46}$$

5.1.3 Examples

Example 5.1.5

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases} \tag{5.1.47}$$

Let $a \in \mathbb{R}$. Since every open interval contains both rational and irrational numbers, for any $\delta > 0$ and any choice of $L \in \mathbb{R}$, there will exist $x \in (a-\delta, a+\delta)$, $x \neq a$, such that

$$|f(x) - L| \geq \frac{1}{2}. \tag{5.1.48}$$

Hence $\lim_{x\to a} f(x)$ does not exist for any real number a.

Example 5.1.6

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases} \tag{5.1.49}$$

Then $\lim_{x\to 0} f(x) = 0$ since, given $\epsilon > 0, |f(x)| < \epsilon$ provided $|x| < \epsilon$.

Exercise 5.1.9

Show that if f is as given in the previous example and $a \neq 0$, then $\lim_{x\to a} f(x)$ does not exist.

Exercise 5.1.11

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational,} \end{cases} \tag{5.1.50}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that, for any real number $a, \lim_{x\to a} f(x) = 0$.

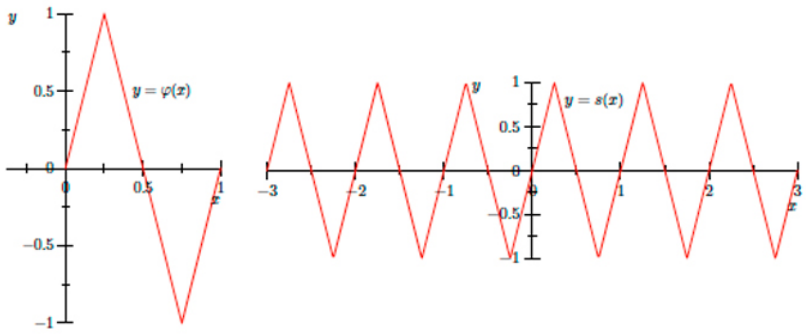

Figure 5.1.1: Graphs of $y = \varphi(x)$ and $y = s(x)$

Example 5.1.7

Define $\varphi : [0, 1] \to [-1, 1]$ by

$$\varphi(x) = \begin{cases} 4x, & \text{if } 0 \leq x \leq \frac{1}{4}, \\ 2 - 4x, & \text{if } \frac{1}{4} < x < \frac{3}{4}, \\ 4x - 4, & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases} \tag{5.1.51}$$

Next define $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \varphi(x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x (that is, $\lfloor x \rfloor$ is the floor of x). The function s is an example of a sawtooth function. See the graphs of φ and s in Figure 5.1.1. Note that for any $n \in \mathbb{Z}$,

$$s([n, n + 1]) = [-1, 1]. \tag{5.1.52}$$

Now let $D = \mathbb{R}\backslash\{0\}$ and define $\sigma : D \to \mathbb{R}$ by

$$\sigma(x) = s\left(\frac{1}{x}\right). \tag{5.1.53}$$

See the graph of σ in Figure 5.1.2. Note that for any $n \in \mathbb{Z}^+$,

$$\sigma\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = s([n, n + 1]) = [-1, 1]. \tag{5.1.54}$$

Hence for any $\epsilon > 0, \sigma((0, \epsilon)) = [-1, 1]$, and so $\lim_{x\to 0^+} \sigma(x)$ does not exist. Similarly, neither $\lim_{x\to 0^-} \sigma(x)$ nor $\lim_{x\to 0} \sigma(x)$ exist.

Example 5.1.8

Let s be the sawtooth function of the previous example and let $D = \mathbb{R}\backslash\{0\}$. Define $\psi : D \to \mathbb{R}$ by

$$\psi(x) = xs\left(\frac{1}{x}\right). \tag{5.1.55}$$

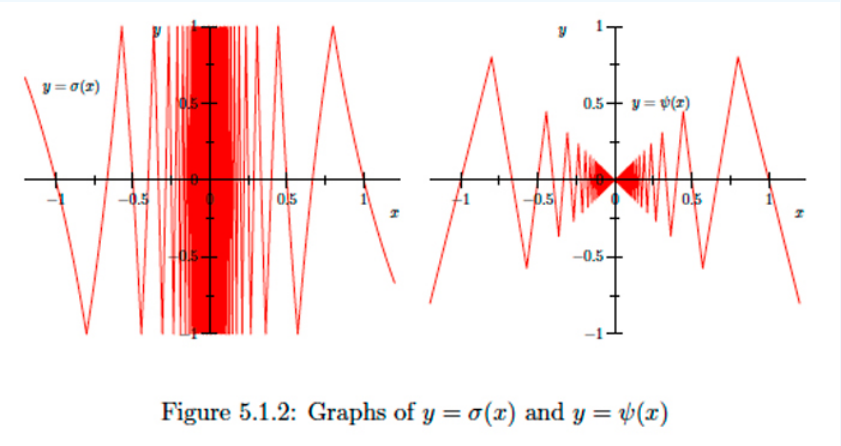

Figure 5.1.2: Graphs of $y = \sigma(x)$ and $y = \psi(x)$

See Figure 5.1 .2 for the graph of ψ. Then for all $x \in D$,

$$-|x| \leq \psi(x) \leq |x|, \tag{5.1.56}$$

and so $\lim_{x\to 0} \psi(x) = 0$ by the squeeze theorem.

Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is bounded if there exists a real number B such that $|f(x)| \leq B$ for all $x \in D$.

Exercise 5.1.12

Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded. Show that $\lim_{x\to 0} xf(x) = 0$.

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5.2: Monotonic Functions

Definition

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $(a, b) \subset D$. We say f is increasing on (a, b) if $f(x) < f(y)$ whenever $a < x < y < b$; we say f is decreasing on (a, b) if $f(x) > f(y)$ whenever $a < x < y < b$; we say f is nondecreasing on (a, b) if $f(x) \leq f(y)$ whenever $a < x < y < b$; and we say f is nonincreasing on (a, b) if $f(x) \geq f(y)$ whenever $a < x < y < b$. We will say f is monotonic on (a, b) if f is either nondecreasing or nonincreasing on (a, b) and we will say f is strictly monotonic on (a, b) if f is either increasing or decreasing on (a, b).

Proposition 5.2.1

If f is monotonic on (a, b), then $f(c+)$ and $f(c-)$ exist for every $c \in (a, b)$.

Proof

Suppose f is nondecreasing on (a, b). Let $c \in (a, b)$ and let

$$\lambda = \sup\{f(x) : a < x < c\}. \tag{5.2.1}$$

Note that $\lambda \leq f(c) < +\infty$. Given any $\epsilon > 0$, there must exist $\delta > 0$ such that

$$\lambda - \epsilon < f(c - \delta) \leq \lambda. \tag{5.2.2}$$

Since f is nondecreasing, it follows that

$$|f(x) - \lambda| < \epsilon \tag{5.2.3}$$

whenever $x \in (c - \delta, c)$. Thus $f(c-) = \lambda$. A similar argument shows that $f(c+) = \kappa$ where

$$\kappa = \inf\{f(x) : c < x < b\}. \tag{5.2.4}$$

If f is nonincreasing, similar arguments yield

$$f(c-) = \inf\{f(x) : a < x < c\} \tag{5.2.5}$$

and

$$f(c+) = \sup\{f(x) : c < x < b\}. \tag{5.2.6}$$

Proposition 5.2.2

If f is nondecreasing on (a, b) and $a < x < y < b$, then

$$f(x+) \leq f(y-). \tag{5.2.7}$$

Proof

By the previous proposition,

$$f(x+) = \inf\{f(t) : x < t < b\} \tag{5.2.8}$$

and

$$f(y-) = \sup\{f(t) : a < t < y\}. \tag{5.2.9}$$

Since f is nondecreasing,

$$\inf\{f(t) : x < t < b\} = \inf\{f(t) : x < t < y\} \tag{5.2.10}$$

and

$$\sup\{f(t) : a < t < y\} = \sup\{f(t) : x < t < y\}. \tag{5.2.11}$$

Thus

$$f(x+) = \inf\{f(t) : x < t < y\} \leq \sup\{f(t) : x < t < y\} = f(y-). \tag{5.2.12}$$

Q.E.D.

Exercise 5.2.1

Let $\varphi : \mathbb{Q} \cap [0,1] \to \mathbb{Z}^+$ be a one-to-one correspondence. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{q \in \mathbb{Q} \cap [0,1]_{q \leq x}} \frac{1}{2^{\varphi(q)}}. \tag{5.2.13}$$

a. Show that f is increasing on $(0,1)$.

b. Show that for any $x \in \mathbb{Q} \cap (0,1)$, $f(x-) < f(x)$ and $f(x+) = f(x)$.

c. Show that for any irrational $a, 0 < a < 1, \lim_{x \to a} f(x) = f(a)$.

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5.3: Limits to Infinity and Infinite Limits

Definition

Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and suppose a is a limit point of D. We say that f diverges to $+\infty$ as x approaches a, denoted

$$\lim_{x \to a} f(x) = +\infty, \tag{5.3.1}$$

if for every real number M there exists a $\delta > 0$ such that

$$f(x) > M \text{ whenever } x \neq a \text{ and } x \in (a - \delta, a + \delta) \cap D. \tag{5.3.2}$$

Similarly, we say that that f diverges to $-\infty$ as x approaches a, denoted

$$\lim_{x \to a} f(x) = -\infty, \tag{5.3.3}$$

if for every real number M there exists a $\delta > 0$ such that

$$f(x) < M \text{ whenever } x \neq a \text{ and } x \in (a - \delta, a + \delta) \cap D. \tag{5.3.4}$$

Exercise 5.3.1

Provide definitions for

a. $\lim_{x \to a^+} f(x) = +\infty$,

b. $\lim_{x \to a^-} f(x) = +\infty$,

c. $\lim_{x \to a^+} f(x) = -\infty$,

d. $\lim_{x \to a^-} f(x) = -\infty$.

Model your definitions on the preceding definitions.

Exercise 5.3.2

Show that $\lim_{x \to 4^+} \frac{7}{4-x} = -\infty$ and $\lim_{x \to 4^-} \frac{7}{4-x} = +\infty$.

Definition

Suppose $D \subset \mathbb{R}$ does not have an upper bound, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that the limit of f as x approaches $+\infty$ is L, denoted

$$\lim_{x \to +\infty} f(x) = L, \tag{5.3.5}$$

if for every $\epsilon > 0$ there exists a real number M such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (M, +\infty) \cap D. \tag{5.3.6}$$

Definition

Suppose $D \subset \mathbb{R}$ does not have an lower bound, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that the limit of f as x approaches $-\infty$ is L, denoted

$$\lim_{x \to -\infty} f(x) = L, \tag{5.3.7}$$

if for every $\epsilon > 0$ there exists a real number M such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (-\infty, M) \cap D. \tag{5.3.8}$$

Exercise 5.3.3

Verify that $\lim_{x\to+\infty} \frac{x+1}{x+2} = 1$.

Exercise 5.3.4

Provide definitions for

a. $\lim_{x\to+\infty} f(x) = +\infty$,

b. $\lim_{x\to+\infty} f(x) = -\infty$,

c. $\lim_{x\to-\infty} f(x) = +\infty$,

d. $\lim_{x\to-\infty} f(x) = -\infty$.

Model your definitions on the preceding definitions.

Exercise 5.3.5

Suppose

$$f(x) = ax^3 + bx^2 + cx + d, \tag{5.3.9}$$

where $a, b, c, d \in \mathbb{R}$ and $a > 0$. Show that

$$\lim_{x\to+\infty} f(x) = +\infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty. \tag{5.3.10}$$

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5.4: Continuous Functions

5.4.1 Continuity at a Point

Definition

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. We say f is continuous at a if either a is an isolated point of D or $\lim_{x\to a} f(x) = f(a)$. If f is not continuous at a, we say f is discontinuous at a, or that f has a discontinuity at a.

Example 5.4.1

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases} \tag{5.4.1}$$

Then, by Example 5.1.5, f is discontinuous at every $x \in \mathbb{R}$.

Example 5.4.2

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases} \tag{5.4.2}$$

Then, by Example 5.1 .6 and Exercise 5.1.10, f is continuous at 0, but discontinuous at every $x \neq 0$.

If $D \subset \mathbb{R}, \alpha \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$, then we define $\alpha f : D \to \mathbb{R}$ by

$$(\alpha f)(x) = \alpha f(x), \tag{5.4.3}$$

$f + g : D \to \mathbb{R}$by

$$(f + g)(x) = f(x) + g(x), \tag{5.4.4}$$

and $fg : D \to \mathbb{R}$ by

$$(fg)(x) = f(x)g(x). \tag{5.4.5}$$

Moreover, if $g(x) \neq 0$ for all $x \in D$, we define $\frac{f}{g} : D \to \mathbb{R}$ by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}. \tag{5.4.6}$$

Proposition 5.4.1

Suppose $D \subset \mathbb{R}, \alpha \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If f and g are continuous at a, then $\alpha f, f + g$, and fg are all continuous at a. Moreover, if $g(x) \neq 0$ for all $x \in D$, then $\frac{f}{g}$ is continuous at a.

Exercise 5.4.1

Prove the previous proposition.

Proposition 5.4.2

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $f(x) \geq 0$ for all $x \in D$, and f is continuous at $a \in D$. If $g : D \to \mathbb{R}$ is defined by $g(x) = \sqrt{f(x)}$, then g is continuous at a.

Exercise 5.4.2

Prove the previous proposition.

Proposition 5.4.3

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. Then f is continuous at a if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in (a - \delta, a + \delta) \cap D. \tag{5.4.7}$$

Proof

Suppose f is continuous at a. If a is an isolated point of D, then there exists a $\delta > 0$ such that

$$(a - \delta, a + \delta) \cap D = \{a\}. \tag{5.4.8}$$

Then for any $\epsilon > 0$, if $x \in (a - \delta, a + \delta) \cap D$, then $x = a$, and so

$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \epsilon. \tag{5.4.9}$$

If a is a limit point of D, then $\lim_{x \to a} f(x) = f(a)$ implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in (a - \delta, a + \delta) \cap D. \tag{5.4.10}$$

Now suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in (a - \delta, a + \delta) \cap D. \tag{5.4.11}$$

If a is an isolated point, then f is continuous at a. If a is a limit point, then this condition implies $\lim_{x \to a} f(x) = f(a)$, and so f is continuous at a. Q.E.D.

From the preceding, it should be clear that a function $f : D \to \mathbb{R}$ is continuous at a point a of D if and only if for every sequence $\{x_n\}_{n \in I}$ with $x_n \in D$ for every $n \in I$ and $\lim_{n \to \infty} x_n = a, \lim_{n \to \infty} f(x_n) = f(a)$.

Exercise 5.4.1

Show that if $f : D \to \mathbb{R}$ is continuous at $a \in D$ and $f(a) > 0$, then there exists an open interval I such that $a \in I$ and $f(x) > 0$ for every $x \in I \cap D$.

Proposition 5.4.4

Suppose $D \subset \mathbb{R}, E \subset \mathbb{R}, g : D \to \mathbb{R}, f : E \to \mathbb{R}, g(D) \subset E$ and $a \in D$. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof

Let $\{x_n\}_{n \in I}$ be a sequence with $x_n \in D$ for every $n \in I$ and $\lim_{n \to \infty} x_n = a$. Then, since g is continuous at $a, \{g(x_n)\}_{n \in I}$ is a sequence with $g(x_n) \in E$ for every $n \in I$ and $\lim_{n \to \infty} g(x_n) = g(a)$. Hence, since f is continuous at $g(a)$, $\lim_{n \to \infty} f(g(x_n)) = f(g(a))$. That is,

$$\lim_{n \to \infty} (f \circ g)(x_n) = (f \circ g)(a). \tag{5.4.12}$$

Hence $f \circ g$ is continuous at a.

Definition

Let $D \subset \mathbb{R}, f : D \to \mathbb{R}$, and $a \in D$. If f is not continuous at a but both $f(a-)$ and $f(a+)$ exist, then we say f has a simple discontinuity at a.

Proposition 5.4.5

Suppose f is monotonic on the interval (a,b). Then every discontinuity of f in (a,b) is a simple discontinuity. Moreover, if E is the set of points in (a,b) at which f is discontinuous, then either $E=\emptyset$, E is finite, or E is countable.

Proof

The first statement follows immediately from Proposition 5.2.1. For the second statement, suppose f is nondecreasing and suppose E is nonempty. From Exercise 2.1 .26 and the the proof of Proposition 5.2.1, it follows that for every $x \in (a,b)$,

$$f(x-) \le f(x) \le f(x+). \tag{5.4.13}$$

Hence $x \in E$ if and only if $f(x-) < f(x+)$. Hence for every $x \in E$, we may choose a rational number r_x such that $f(x-) < r_x < f(x+)$. Now if $x, y \in E$ with $x < y$, then, by Proposition 5.2.2,

$$r_x < f(x+) \le f(y-) < r_y, \tag{5.4.14}$$

so $r_x \neq r_y$. Thus we have a one-to-one correspondence between E and a subset of $\mathbb{Q}$, and so E is either finite or countable. A similar argument holds if f is nonincreasing. Q.E.D.

Exercise 5.4.4

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational.} \end{cases} \tag{5.4.15}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that f is continuous at every irrational number and has a simple discontinuity at every rational number.

5.4.2 Continuity on a Set

Definition

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is continuous on D if f is continuous at every point $a \in D$.

Proposition 5.4.6

If f is a polynomial, then f is continuous on $\mathbb{R}$.

Proposition 5.4.7

If $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a rational function, then f is continuous on D.

Exercise 5.4.5

Explain why the function $f(x) = \sqrt{1-x^2}$ is continuous on $[-1,1]$.

Exercise 5.4.6

Discuss the continuity of the function

$$f(x) = \begin{cases} x+1, & \text{if } x < 0, \\ 4, & \text{if } x = 0, \\ x^2, & \text{if } x > 0. \end{cases} \tag{5.4.16}$$

If $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $E \subset \mathbb{R}$, we let

$$f^{-1}(E) = \{x \in D : f(x) \in E\} \tag{5.4.17}$$

$$f^{-1}(E) = \{x : f(x) \in E\}. \tag{5.4.17}$$

Proposition 5.4.8

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if for every open set $V \subset \mathbb{R}$, $f^{-1}(V) = U \cap D$ for some open set $U \subset \mathbb{R}$.

Proof

Suppose f is continuous on D and $V \subset \mathbb{R}$ is an open set. If $V \cap f(D) = \emptyset$, then $f^{-1}(V) = \emptyset$, which is open. So suppose $V \cap f(D) \neq \emptyset$ and let $a \in f^{-1}(V)$. Since V is open and $f(a) \in V$, there exists $\epsilon_a > 0$ such that

$$(f(a) - \epsilon_a, f(a) + \epsilon_a) \subset V. \tag{5.4.18}$$

Since f is continuous, there exists $\delta_a > 0$ such that

$$f((a - \delta_a, a + \delta_a) \cap D) \subset (f(a) - \epsilon_a, f(a) + \epsilon_a) \subset V. \tag{5.4.19}$$

That is, $(a - \delta_a, a + \delta_a) \cap D \subset f^{-1}(V)$. Let

$$U = \bigcup_{a \in f^{-1}(V)} (a - \delta_a, a + \delta_a). \tag{5.4.20}$$

Then U is open and $f^{-1}(V) = U \cap D$.

Now suppose that for every open set $V \subset \mathbb{R}$, $f^{-1}(V) = U \cap D$ for some open set $U \subset \mathbb{R}$. Let $a \in D$ and let $\epsilon > 0$ be given. Since $(f(a) - \epsilon, f(a) + \epsilon)$ is open, there exists an open set U such that

$$U \cap D = f^{-1}((f(a) - \epsilon, f(a) + \epsilon)). \tag{5.4.21}$$

Since U is open and $a \in U$, there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset U$. But then

$$f((a - \delta, a + \delta) \cap D) \subset (f(a) - \epsilon, f(a) + \epsilon). \tag{5.4.22}$$

That is, if $x \in (a - \delta, a + \delta) \cap D$, then $|f(x) - f(a)| < \epsilon$. Hence f is continuous at a. Q.E.D.

Exercise 5.4.7

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. For any $E \subset \mathbb{R}$, show that $f^{-1}(\mathbb{R} \backslash E) = (\mathbb{R} \backslash f^{-1}(E)) \cap D$.

Exercise 5.4.8

Let A be a set and, for each $\alpha \in A$, let $U_\alpha \subset \mathbb{R}$. Given $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$, show that

$$\bigcup_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) \tag{5.4.23}$$

and

$$\bigcap_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right). \tag{5.4.24}$$

Exercise 5.4.9

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Show that f is continuous on D if and only if for every closed set $C \subset \mathbb{R}$, $f^{-1}(C) = F \cap D$ for some closed set $F \subset \mathbb{R}$.

Exercise 5.4.10

Let $D \subset \mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is Lipschitz if there exists $\alpha \in \mathbb{R}, \alpha > 0$, such that $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in D$. Show that if f is Lipschitz, then f is continuous.

5.4.3 Intermediate Value Theorem

Theorem 5.4.9

(Intermediate Value Theorem).

Suppose $a, b \in \mathbb{R}, a < b$, and $f : [a, b] \to \mathbb{R}$. If f is continuous and $s \in \mathbb{R}$ is such that either $f(a) \leq s \leq f(b)$ or $f(b) \leq s \leq f(a)$, then there exists $c \in [a, b]$ such that $f(c) = s$.

Proof

Suppose $f(a) < f(b)$ and $f(a) < s < f(b)$. Let

$$c = \sup\{x : x \in [a, b], f(x) \leq s\}. \tag{5.4.25}$$

Suppose $f(c) < s$. Then $c < b$ and, since f is continuous at c, there exists a $\delta > 0$ such that $f(x) < s$ for all $x \in (c, c + \delta)$. But then $f\left(c + \frac{\delta}{2}\right) < s$, contradicting the definition of c. Similarly, if $f(c) > s$, then $c > a$ and there exists $\delta > 0$ such that $f(x) > s$ for all $x \in (c - \delta, c)$, again contradicting the definition of c. Hence we must have $f(c) = s$. Q.E.D.

Example 5.4.3

Suppose $a \in \mathbb{R}, a > 0$, and consider $f(x) = x^n - a$ where $n \in \mathbb{Z}, n > 1$. Then $f(0) = -a < 0$ and

$$\begin{aligned} f(1 + a) &= (1 + a)^n - a \\ &= 1 + na + \sum_{i=2}^{n} \binom{n}{i} a^i - a \\ &= 1 + (n - 1)a + \sum_{i=2}^{n} \binom{n}{i} a^i > 0, \end{aligned} \tag{5.4.26}$$

where $\binom{n}{i}$ is the binomial coefficient

$$\binom{n}{i} = \frac{n!}{i!(n - i)!}. \tag{5.4.27}$$

Hence, by the Intermediate Value Theorem, there exists a real number $\gamma > 0$ such that $\gamma^n = a$. Moreover, there is only one such γ since f is increasing on $(0, +\infty)$.

We call γ the n th root of a, and write

$$\gamma = \sqrt[n]{a} \tag{5.4.28}$$

or

$$\gamma = a^{\frac{1}{n}}. \tag{5.4.29}$$

Moreover, if $a \in \mathbb{R}, a < 0, n \in Z^+$ is odd, and γ is the nth root of $-a$, then

$$(-\gamma)^n = (-1)^n(\gamma)^n = (-1)(-a) = a. \tag{5.4.30}$$

That is, $-\gamma$ is the n th root of a.

Definition

If $n = \frac{p}{q} \in \mathbb{Q}$ with $q \in \mathbb{Z}^+$, then we define

$$x^n = (\sqrt[q]{x})^p \tag{5.4.31}$$

for all real $x \geq 0$.

Exercise 5.4.11

Explain why the equation $x^5 + 4x^2 - 16 = 0$ has a solution in the interval $(0, 2)$.

Exercise 5.4.12

Give an example of a closed interval $[a, b] \subset \mathbb{R}$ and a function $f : [a, b] \to \mathbb{R}$ which do not satisfy the conclusion of the Intermediate Value Theorem.

Exercise 5.4.13

Show that if $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.

Exercise 5.4.14

Suppose $f : (a, b) \to \mathbb{R}$ is continuous and strictly monotonic. Let $(c, d) = f((a, b))$. Show that $f^{-1} : (c, d) \to (a, b)$ is strictly monotonic and continuous.

Exercise 5.4.15

Let $n \in \mathbb{Z}^+$. Show that the function $f(x) = \sqrt[n]{x}$ is continuous on $(0, +\infty)$.

Exercise 5.4.16

Use the method of bisection to give another proof of the Intermediate Value Theorem.

5.4.4 Extreme Value Theorem

Theorem 5.4.10

Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Proof

Given a sequence $\{y_n\}_{n \in I}$ in $f(D)$, choose a sequence $\{x_n\}_{n \in I}$ such that $f(x_n) = y_n$. Since D is compact, $\{x_n\}_{n \in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} x_{n_k} = x \in D. \tag{5.4.32}$$

Let $y = f(x)$. Then $y \in f(D)$ and, since f is continuous,

$$y = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} y_{n_k}. \tag{5.4.33}$$

Hence $f(D)$ is compact.

Exercise 5.4.17

Prove the previous theorem using the open cover definition of a compact set.

Theorem 5.4.11

(Extreme Value Theorem).

Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then there exists $a \in D$ such that $f(a) \geq f(x)$ for all $x \in D$ and there exists $b \in D$ such that $f(b) \leq f(x)$ for all $x \in D$. Q.E.D.

As a consequence of the Extreme Value Theorem, a continuous function on a closed bounded interval attains both a maximum and a minimum value.

Exercise 5.4.18

Find an example of a closed bounded interval $[a, b]$ and a function $f : [a, b] \to \mathbb{R}$ such that f attains neither a maximum nor a minimum value on $[a, b]$.

Exercise 5.4.19

Find an example of a bounded interval I and a function $f : I \to \mathbb{R}$ which is continuous on I such that f attains neither a maximum nor a minimum value on I.

Exercise 5.4.20

Suppose $K \subset \mathbb{R}$ is compact and $a \notin K$. Show that there exists $b \in K$ such that $|b - a| \leq |x - a|$ for all $x \in K$.

Proposition 5.4.12

Suppose $D \subset \mathbb{R}$ is compact, $f : D \to \mathbb{R}$ is continuous and one-to-one, and $E = f(D)$. Then $f^{-1} : E \to D$ is continuous.

Proof

Let $V \subset \mathbb{R}$ be an open set. We need to show that $f(V \cap D) = U \cap E$ for some open set $U \subset \mathbb{R}$. Let $C = D \cap (\mathbb{R} \backslash V)$. Then C is a closed subset of D, and so is compact. Hence $f(C)$ is a compact subset of E. Thus $f(C)$ is closed, and so $U = \mathbb{R} \backslash f(C)$ is open. Moreover, $U \cap E = E \backslash f(C) = f(V \cap D)$. Thus f^{-1} is continuous.

Exercise 5.4.21

Suppose $f : [0, 1] \cup (2, 3] \to [0, 2]$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } 2 < x \leq 3. \end{cases} \tag{5.4.34}$$

Show that f is continuous, one-to-one, and onto, but that f^{-1} is not continuous.

5.4.5 Uniform Continuity

Definition

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is uniformly continuous on D if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in D$,

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta. \tag{5.4.35}$$

Exercise 5.4.22

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is Lipschitz (see Exercise 5.4.10). Show that f is uniformly continuous on D.

Clearly, if f is uniformly continuous on D then f is continuous on D. However, a continuous function need not be uniformly continuous.

Example 5.4.4

Define $f : (0, +\infty)$ by $f(x) = \frac{1}{x}$, Given any $\delta > 0$, choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n(n+1)} < \delta$. Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Then

$$|x - y| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \delta. \tag{5.4.36}$$

However,

$$|f(x) - f(y)| = |n - (n+1)| = 1. \tag{5.4.37}$$

Hence, for example, there does not exist a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{1}{2} \tag{5.4.38}$$

whenever $|x - y| < \delta$. Thus f is not uniformly continuous on $(0, +\infty)$, although f is continuous on $(0, +\infty)$.

Example 5.4.5

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$. Let $\epsilon > 0$ be given. If $\delta = \frac{\epsilon}{2}$, then

$$|f(x) - f(y)| = 2|x - y| < \epsilon \tag{5.4.39}$$

whenever $|x - y| < \delta$. Hence f is uniformly continuous on $\mathbb{R}$.

Exercise 5.4.23

Let $f(x) = x^2$. Show that f is not uniformly continuous on $(-\infty, +\infty)$.

Proposition 5.4.13

Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then f is uniformly continuous on D.

Proof

Let $\epsilon > 0$ be given. For every $x \in D$, choose δ_x such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \tag{5.4.40}$$

whenever $y \in D$ and $|x - y| < \delta_x$. Let

$$J_x = \left(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}\right). \tag{5.4.41}$$

Then $\{J_x : x \in D\}$ is an open cover of D. Since D is compact, there must exist $x_1, x_2, \ldots, x_n, n \in Z^+$, such that $J_{x_1}, J_{x_2}, \ldots, J_{x_n}$ is an open cover of D. Let δ be the smallest of

$$\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_n}}{2}. \tag{5.4.42}$$

Now let $x, y \in D$ with $|x - y| < \delta$. Then for some integer $k, 1 \leq k \leq n, x \in J_{x_k}$, that is,

$$|x - x_k| < \frac{\delta_{x_k}}{2}. \tag{5.4.43}$$

Moreover,

$$|y - x_k| \leq |y - x| + |x - x_k| < \delta + \frac{\delta_{x_k}}{2} \leq \delta_{x_k}. \tag{5.4.44}$$

Hence

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{5.4.45}$$

Q.E.D.

Exercise 5.4.24

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is uniformly continuous. Show that if $\{x_n\}_{n \in I}$ is a Cauchy sequence in D, then $\{f(x_n)\}_{n \in I}$ is a Cauchy sequence in $f(D)$.

Exercise 5.4.25

Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Show that $f(0+)$ exists.

Exercise 5.4.26

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = 0$. Show that f is uniformly continuous.

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CHAPTER OVERVIEW

6: Derivatives

6.1: Best Linear Approximations
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Thumbnail: Derivatives (CC BY; OpenStax)

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6.1: Best Linear Approximations

Definition

We say a function $f : \mathbb{R} \to \mathbb{R}$ is linear if for every $x, y \in \mathbb{R}$,

$$f(x + y) = f(x) + f(y) \tag{6.1.1}$$

and for every $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$f(\alpha x) = \alpha f(x). \tag{6.1.2}$$

Exercise 6.1.1

Show that if $f : \mathbb{R} \to \mathbb{R}$ is linear, then there exists $m \in \mathbb{R}$ such that $f(x) = mx$ for all $x \in \mathbb{R}$.

Definition

Suppose $D \in \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D. We say f is differentiable at a if there exists a linear function $df_a : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - df_a(x - a)}{x - a} = 0. \tag{6.1.3}$$

We call the function df_a the best linear approximation to f at a, or the differential of f at a.

Proposition 6.1.1

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D. Then f is differentiable at a if and only if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{6.1.4}$$

exists, in which case $df_a(x) = mx$ where

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{6.1.5}$$

Proof

Let $m \in \mathbb{R}$ and let $L : \mathbb{R} \to \mathbb{R}$ be the linear function $L(x) = mx$. Then

$$\begin{aligned} \frac{f(x) - f(a) - L(x - a)}{x - a} &= \frac{f(x) - f(a) - m(x - a)}{x - a} \\ &= \frac{f(x) - f(a)}{x - a} - m. \end{aligned} \tag{6.1.6}$$

Hence

$$\lim_{x \to a} \frac{f(x) - f(a) - L(x - a)}{x - a} = 0 \tag{6.1.7}$$

if and only if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m. \tag{6.1.8}$$

Q.E.D.

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6.2: Derivatives

Definition

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, a is an interior point of D, and f is differentiable at a. We call

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{6.2.1}$$

the derivative of f at a, which we denote $f'(a)$.

Note that if f is differentiable at a, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{6.2.2}$$

Definition

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and E is the set of interior points of D at which f is differentiable. We call the function $f' : E \to \mathbb{R}$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{6.2.3}$$

the derivative of f.

Example 6.2.1

Let $n \in \mathbb{Z}^+$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x + h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \sum_{k=2}^{n} \binom{n}{k} x^{n-k}h^k - x^n}{h} \\ &= \lim_{h \to 0} \left(nx^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x^{n-k}h^{k-1} \right) \\ &= nx^{n-1}. \end{aligned} \tag{6.2.4}$$

Example 6.2.2

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Then

$$\frac{f(0 + h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1, \text{ if } h > 0, \\ -1, \text{ if } h < 0. \end{cases} \tag{6.2.5}$$

Hence

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = -1 \tag{6.2.6}$$

and

$$\lim_{h \to 0+} \frac{f(0 + h) - f(0)}{h} = 1. \tag{6.2.7}$$

Thus f is not differentiable at 0.

Exercise 6.2.1

Show that if $c \in \mathbb{R}$ and $f(x) = c$ for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 6.2.2

Define $f : [0, +\infty) \to [0, +\infty)$ by $f(x) = \sqrt{x}$. Show that $f' : (0, +\infty) \to (0, +\infty)$ is given by

$$f'(x) = \frac{1}{2\sqrt{x}}. \tag{6.2.8}$$

Exercise 6.2.3

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ x^2, & \text{if } x \geq 0. \end{cases} \tag{6.2.9}$$

Is f differentiable at 0?

Exercise 6.2.4

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x^3, & \text{if } x \geq 0. \end{cases} \tag{6.2.10}$$

Is f differentiable at 0?

Proposition 6.2.1

If f is differentiable at a, then f is continuous at a.

Proof

If f is differentiable at a, then

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)(x - a) = f'(a)(0) = 0. \tag{6.2.11}$$

Hence $\lim_{x \to a} f(x) = f(a)$, and so f is continuous at a. Q.E.D.

6.2.1 The Rules

Proposition 6.2.2

Suppose f is differentiable at a and $\alpha \in \mathbb{R}$. Then αf is differentiable at a and $(\alpha f)'(a) = \alpha f'(a)$.

Exercise 6.2.5

Prove the previous proposition.

Proposition 6.2.3

Suppose f and g are both differentiable at a. Then $f + g$ is differentiable at a and $(f + g)'(a) = f'(a) + g'(a)$.

Exercise 6.2.6

Prove the previous proposition.

Proposition 6.2.4

(Product rule).

Suppose f and g are both differentiable at a. Then fg is differentiable at a and

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a). \tag{6.2.12}$$

Proof

We have

$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \left(g(a+h)\frac{f(a+h) - f(a)}{h} + f(a)\frac{g(a+h) - g(a)}{h} \right) \\ &= g(a)f'(a) + f(a)g'(a).\end{aligned} \tag{6.2.13}$$

where we know $\lim_{h\to 0} g(a+h) = g(a)$ by the continuity of g at a, which in turn follows from the assumption that g is differentiable at a. Q.E.D.

Exercise 6.2.7

Given $n \in \mathbb{Z}^+$ and $f(x) = x^n$, use induction and the product rule to show that $f'(x) = nx^{n-1}$.

Proposition 6.2.5

(Quotient rule).

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, a is in the interior of D, and $g(x) \neq 0$ for all $x \in D$. If f and g are both differentiable at a, then $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}. \tag{6.2.14}$$

Proof

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \lim_{h\to 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a+h)}{hg(a+h)g(a)} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(a+h)}{hg(a+h)g(a)} \\ &= \lim_{h\to 0} \frac{g(a)\frac{f(a+h)-f(a)}{h} - f(a)\frac{g(a+h)-g(a)}{h}}{g(a+h)g(a)} \\ &= \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2};\end{aligned} \tag{6.2.15}$$

where we know $\lim_{h\to 0} g(a+h) = g(a)$ by the continuity of g at a, which in turn follows from the assumption that g is differentiable at a. Q.E.D.

Exercise 6.2.8

Show that for any integer $n \neq 0$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proposition 6.2.6

(Chain rule).

Suppose $D \subset \mathbb{R}, E \subset \mathbb{R}, g : D \to \mathbb{R}, f : E \to \mathbb{R}, g(D) \subset E, g$ is differentiable at a, and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = f'(g(a))g'(a). \tag{6.2.16}$$

Proof

Since a is an interior point of D and $g(a)$ is an interior point of E, we may choose $\delta > 0$ so that $(a - \delta, a + \delta) \subset D$ and $\epsilon > 0$ so that $(g(a) - \epsilon, g(a) + \epsilon) \subset E$. Define $\varphi : (-\delta, \delta) \to \mathbb{R}$ by

$$\varphi(h) = \begin{cases} \frac{g(a+h)-g(a)-g'(a)h}{h}, & \text{if } h \neq 0, \\ 0, & \text{if } h = 0, \end{cases} \tag{6.2.17}$$

and $\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$\psi(h) = \begin{cases} \frac{f(g(a)+h)-f(g(a))-f'(g(a))h}{h}, & \text{if } h \neq 0, \\ 0, & \text{if } h = 0. \end{cases} \tag{6.2.18}$$

The assumption that g is differentiable at a implies that φ is continuous at 0 and the assumption that f is differentiable at $g(a)$ implies that ψ is continuous at 0. Moreover, note that

$$g(a + h) = h\varphi(h) + g'(a)h + g(a) \tag{6.2.19}$$

for $h \in (-\delta, \delta)$ and

$$f(g(a) + h) = h\psi(h) + f'(g(a))h + f(g(a)) \tag{6.2.20}$$

for $h \in (-\epsilon, \epsilon)$. From (6.2.12) we have

$$f(g(a + h)) = f\left(h\varphi(h) + g'(a)h + g(a)\right) \tag{6.2.21}$$

for $h \in (-\delta, \delta)$. Now

$$\lim_{h\to 0} \left(h\varphi(h) + g'(a)h\right) = 0, \tag{6.2.22}$$

so we may choose $\gamma > 0$ so that $\gamma \leq \delta$ and

$$|h\varphi(h) + g'(a)h| < \epsilon \tag{6.2.23}$$

whenever $h \in (-\gamma, \gamma)$. Thus, using (6.2.13) and (6.2.14),

$$f(g(a + h)) = \left(h\varphi(h) + g'(a)h\right)\psi\left(h\varphi(h) + g'(a)h\right) + f'(g(a))\left(h\varphi(h) + g'(a)h\right) + f(g(a)), \tag{6.2.24}$$

so

$$\begin{aligned} f(g(a + h)) - f(g(a)) =& \left(h\varphi(h) + g'(a)h\right)\psi\left(h\varphi(h) + g'(a)h\right) \\ &+ f'(g(a))\left(h\varphi(h) + g'(a)h\right) \\ =& h\varphi(h)\psi\left(h\varphi(h) + g'(a)h\right) \\ &+ hg'(a)\psi\left(h\varphi(h) + g'(a)h\right) \\ &+ f'(g(a))\varphi(h)h + f'(g(a))g'(a)h. \end{aligned} \tag{6.2.25}$$

Hence

$$\begin{aligned} \frac{f(g(a + h)) - f(g(a))}{h} =& f'(g(a))g'(a) + \varphi(h)\psi\left(h\varphi(h) + g'(a)h\right) \\ &+ g'(a)\psi\left(h\varphi(h) + g'(a)h\right) + f'(g(a))\varphi(h). \end{aligned} \tag{6.2.26}$$

Now

$$\lim_{h\to 0}\varphi(h)=0, \tag{6.2.27}$$

$$\lim_{h\to 0}\left(h\varphi(h)+g'(a)h\right)=0, \tag{6.2.28}$$

and, since φ and ψ are continuous at 0,

$$\lim_{h\to 0}\psi\left(h\varphi(h)+g'(a)h\right)=0. \tag{6.2.29}$$

Thus

$$\lim_{h\to 0}\frac{f(g(a+h))-f(g(a))}{h}=f'(g(a))g'(a). \tag{6.2.30}$$

Q.E.D.

Proposition 6.2.7

Suppose $D\subset\mathbb{R}$, $f:D\to\mathbb{R}$ is one-to-one, a is in the interior of D, $f(a)$ is in the interior of $f(D)$, f^{-1} is continuous at $f(a)$, and f is differentiable at a with $f'(a)\neq 0$. Then f^{-1} is differentiable at $f(a)$ and

$$\left(f^{-1}\right)'(f(a))=\frac{1}{f'(a)}. \tag{6.2.31}$$

Proof

Choose $\delta>0$ so that $(f(a)-\delta,f(a)+\delta)\subset f(D)$. For $h\in(-\delta,\delta)$, let

$$k=f^{-1}(f(a)+h)-a. \tag{6.2.32}$$

Then

$$f^{-1}(f(a)+h)=a+k, \tag{6.2.33}$$

so

$$f(a)+h=f(a+k) \tag{6.2.34}$$

and

$$h=f(a+k)-f(a). \tag{6.2.35}$$

Hence

$$\frac{f^{-1}(f(a)+h)-f^{-1}(f(a))}{h}=\frac{a+k-a}{f(a+k)-f(a)}=\frac{1}{\frac{f(a+k)-f(a)}{k}}. \tag{6.2.36}$$

Now if $h\to 0,$ then $k\to 0$ (since f^{-1} is continuous at $f(a)$), and so

$$\lim_{h\to 0}\frac{f^{-1}(f(a)+h)-f^{-1}(f(a))}{h}=\lim_{k\to 0}\frac{1}{\frac{f(a+k)-f(a)}{k}}=\frac{1}{f'(a)}. \tag{6.2.37}$$

Q.E.D.

Example 6.2.3

For $n\in Z^+$, define $f:[0,+\infty)\to\mathbb{R}$ by $f(x)=\sqrt[n]{x}$. Then f is the inverse of $g:[0,+\infty)\to\mathbb{R}$ defined by $g(x)=x^n$. Thus, for any $x\in(0,+\infty)$,

$$f'(x)=\frac{1}{g'(f(x))}=\frac{1}{n(\sqrt[n]{x})^{n-1}}=\frac{1}{n}x^{\frac{1}{n}-1}. \tag{6.2.38}$$

Exercise 6.2.9

Let $n \neq 0$ be a rational number and let $f(x) = x^n$. Show that $f'(x) = nx^{n-1}$.

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6.3: Mean Value Theorem

Definition

We say f is differentiable on an open interval I if f is differentiable at every point $a \in I$.

Definition

Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f has a local maximum at a point $a \in D$ if there exists $\delta > 0$ such that $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta) \cap D$. We say f has a local minimum at a point $a \in D$ if there exists $\delta > 0$ such that $f(a) \leq f(x)$ for all $x \in (a - \delta, a + \delta) \cap D$.

Proposition 6.3.1

Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D at which f has either a local maximum or a local minimum. If f is differentiable at a, then $f'(a) = 0$.

Proof

Suppose f has a local maximum at a (a similar argument works if f has a local minimum at a). Choose $\delta > 0$ so that $(a - \delta, a + \delta) \subset D$ and $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta)$. Then

$$\frac{f(x) - f(a)}{x - a} \geq 0 \tag{6.3.1}$$

for all $x \in (a - \delta, a)$ and

$$\frac{f(x) - f(a)}{x - a} \leq 0 \tag{6.3.2}$$

for all $x \in (a, a + \delta)$. Hence

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0 \tag{6.3.3}$$

and

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0. \tag{6.3.4}$$

Hence

$$0 \leq \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0, \tag{6.3.5}$$

so we must have $f'(a) = 0$. Q.E.D.

Theorem 6.3.2

(Rolle's Theorem).

Let $a, b \in \mathbb{R}$ and suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ at which $f'(c) = 0$.

Proof

By the Extreme Value Theorem, we know f attains a maximum and a minimum value on $[a, b]$. Let m be the minimum value and M the maximum value of f on $[a, b]$. If $m = M = f(a) = f(b)$, then $f(x) = m$ for all $x \in [a, b]$, and so $f'(x) = 0$ for all $x \in (a, b)$. Otherwise, one of m or M occurs at a point c in (a, b). Hence f has either a local maximum or a local minimum at c, and so $f'(c) = 0$. Q.E.D.

Exercise 6.3.1

Suppose f is differentiable on (a,b) and $f'(x) \neq 0$ for all $x \in (a,b)$. Show that for any $x, y \in (a,b)$, $f(x) \neq f(y)$.

Exercise 6.3.2

Explain why the equation $x^5 + 10x = 5$ has exactly one solution.

Exercise 6.3.3

Let $f(x)$ be a third degree polynomial. Show that the equation $f(x) = 0$ as at least one, but no more than three, solutions.

6.3.2 Mean Value Theorem

Theorem 6.3.3

(Generalized Mean Value Theorem).

Let $a, b \in \mathbb{R}$. If f and g are continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ at which

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{6.3.6}$$

Proof

Let

$$h(t) = (f(b) - f(a))g(t) - (g(b) - g(a))f(t). \tag{6.3.7}$$

Then h is continuous on $[a,b]$ and differentiable on (a,b). Moreover,

$$\begin{aligned} h(a) &= f(b)g(a) - f(a)g(a) - f(a)g(b) + f(a)g(a) \\ &= f(b)g(a) - f(a)g(b) \end{aligned} \tag{6.3.8}$$

and

$$\begin{aligned} h(b) &= f(b)g(b) - f(a)g(b) - f(b)g(b) + f(b)g(a) \\ &= f(b)g(a) - f(a)g(b). \end{aligned} \tag{6.3.9}$$

Hence, by Rolle's theorem, there exists a point $c \in (a,b)$ at which $h'(c) = 0$. But then

$$0 = h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c), \tag{6.3.10}$$

which implies that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{6.3.11}$$

Q.E.D.

Theorem 6.3.4

(Mean Value Theorem).

Let $a, b \in \mathbb{R}$. If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(c). \tag{6.3.12}$$

Proof

Apply the previous result with $g(x) = x$. Q.E.D.

Exercise 6.3.4

Prove the Mean Value Theorem using Rolle's theorem and the function

$$k(t) = f(t) - \left(\left(\frac{f(b) - f(a)}{b - a} \right) (t - a) + f(a) \right). \tag{6.3.13}$$

Give a geometric interpretation for k and compare it with the function h used in the proof of the generalized mean value theorem.

Exercise 6.3.5

Let $a, b \in \mathbb{R}$. Suppose f is continuous on $[a, b]$, differentiable on (a, b), and $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that

$$|f(b) - f(a)| \leq M|b - a|. \tag{6.3.14}$$

Exercise 6.3.6

Show that for all $x > 0$,

$$\sqrt{1 + x} < 1 + \frac{x}{2}. \tag{6.3.15}$$

Exercise 6.3.7

Suppose I is an open interval, $f : I \to \mathbb{R}$, and $f'(x) = 0$ for all $x \in I$. Show that there exists $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ for all $x \in I$.

Exercise 6.3.8

Suppose I is an open interval, $f : I \to \mathbb{R}, g : I \to \mathbb{R}$, and $f'(x) = g'(x)$ for all $x \in I$. Show that there exists $\alpha \in \mathbb{R}$ such that

$$g(x) = f(x) + \alpha \tag{6.3.16}$$

for all $x \in I$.

Exercise 6.3.9

Let $D = \mathbb{R} \backslash \{0\}$. Define $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ by $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x^2 + 1, & \text{if } x > 0. \end{cases} \tag{6.3.17}$$

Show that $f'(x) = g'(x)$ for all $x \in D$, but there does not exist $\alpha \in \mathbb{R}$ such that $g(x) = f(x) + \alpha$ for all $x \in D$. Why does this not contradict the conclusion of the previous exercise?

Proposition 6.3.5

If f is differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b).

Proof

Let $x, y \in (a, b)$ with $x < y$. By the Mean Value Theorem, there exists a point $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x) f'(c). \tag{6.3.18}$$

Since $y - x > 0$ and $f'(c) > 0$, we have $f(y) > f(x)$, and so f is increasing on (a, b). Q.E.D.

Theorem 6.3.6

If f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Exercise 6.3.10

State and prove similar conditions for nonincreasing and nondecreasing functions.

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6.4: Discontinuities of Derivatives

Theorem 6.4.1: Intermediate Value Theorem for Derivatives

Suppose f is differentiable on an open interval $I, a, b \in I$, and $a < b$. If $\lambda \in \mathbb{R}$ and either $f'(a) < \lambda < f'(b)$ or $f'(a) > \lambda > f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \lambda$.

Proof

Suppose $f'(a) < \lambda < f'(b)$ and define $g : I \to \mathbb{R}$ by $g(x) = f(x) - \lambda x$. Then g is differentiable on I, and so continuous on $[a, b]$. Let c be a point in $[a, b]$ at which g attains its minimum value. Now

$$g'(a) = f'(a) - \lambda < 0, \tag{6.4.1}$$

so there exists $a < t < b$ such that

$$g(t) - g(a) < 0. \tag{6.4.2}$$

Thus $c \neq a$. Similarly,

$$g'(b) = f'(b) - \lambda > 0, \tag{6.4.3}$$

so there exists $a < s < b$ such that

$$g(s) - g(b) < 0. \tag{6.4.4}$$

Thus $c \neq b$. Hence $c \in (a, b)$, and so $g'(c) = 0$. Thus $0 = f'(c) - \lambda$, and so $f'(c) = \lambda$. Q.E.D.

Exercise 6.4.1

Define $g : (-1, 1) \to \mathbb{R}$ by

$$g(x) = \begin{cases} -1, & \text{if } -1 < x < 0, \\ 1, & \text{if } 0 \leq x < 1. \end{cases} \tag{6.4.5}$$

Does there exist a function $f : (-1, 1) \to \mathbb{R}$ such that $f'(x) = g(x)$ for all $x \in (-1, 1)$?

Exercise 6.4.2

Suppose f is differentiable on an open interval I. Show that f' cannot have any simple discontinuities in I.

Example 6.4.1

Define $\varphi : [0, 1] \to \mathbb{R}$ by $\varphi(x) = x(2x - 1)(x - 1)$. Define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(x) = 6x^2 - 6x + 1$. Then

$$\varphi(x) = 2x^3 - 3x^2 + x, \tag{6.4.6}$$

so $\varphi'(x) = \rho(x)$ for all $x \in (0, 1)$. Next define $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \varphi(x - \lfloor x \rfloor)$. See Figure 6.4.1 for the graphs of φ and s. Then for any $n \in \mathbb{Z}$ and $n < x < n + 1$,

$$s'(x) = \rho(x - n) = \rho(x - \lfloor x \rfloor). \tag{6.4.7}$$

Moreover, if x is an integer,

$$\begin{aligned} \lim_{h \to 0^+} \frac{s(x + h) - s(x)}{h} &= \lim_{h \to 0^+} \frac{\varphi(h)}{h} \\ &= \lim_{h \to 0^+} \frac{h(2h - 1)(h - 1)}{h} \\ &= \lim_{h \to 0^+} (2h - 1)(h - 1) \\ &= 1 \end{aligned} \tag{6.4.8}$$

and

$$\begin{aligned}\lim_{h\to 0^-}\frac{s(x+h)-s(x)}{h} &= \lim_{h\to 0^-}\frac{\varphi(h+1)}{h}\\ &= \lim_{h\to 0^-}\frac{(h+1)(2h+1)h}{h}\\ &= \lim_{h\to 0^-}(h+1)(2h+1)\\ &= 1.\end{aligned} \tag{6.4.9}$$

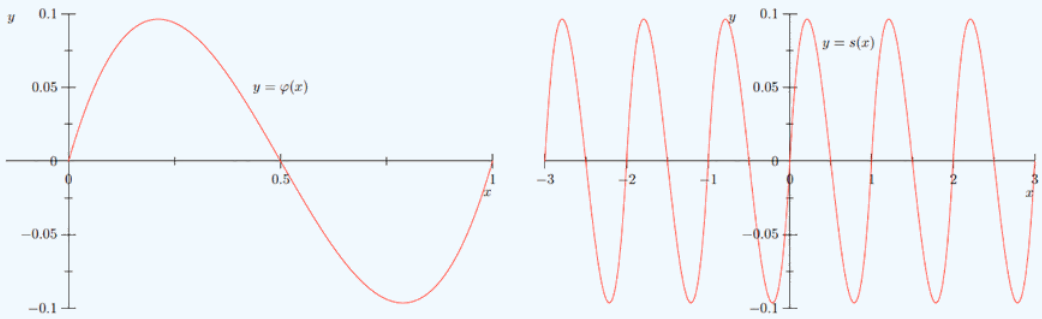

Figure 6.4.1: Graphs of $y=\varphi(x)$ and $y=s(x)$

Thus $s'(x)=1=\rho(x-\lfloor x\rfloor)$ when x is an integer, and so $s'(x)=\rho(x-\lfloor x\rfloor)$ for all $x\in\mathbb{R}$.

Now $\rho(x)=0$ if and only if $x=\frac{3-\sqrt{3}}{6}$ or $x=\frac{3+\sqrt{3}}{6}$. since $\varphi(0)=0$, $\varphi\left(\frac{3-\sqrt{3}}{6}\right)=\frac{1}{6\sqrt{3}}$, $\varphi\left(\frac{3+\sqrt{3}}{6}\right)=-\frac{1}{6\sqrt{3}}$, and $\varphi(1)=0$, we see that φ attains a

maximum value of $\frac{1}{6\sqrt{3}}$ and a minimum value of $-\frac{1}{6\sqrt{3}}$. Hence for any $n\in\mathbb{Z}$,

$$s((n,n+1))=\left[-\frac{1}{6\sqrt{3}},\frac{1}{6\sqrt{3}}\right]. \tag{6.4.10}$$

Also, $\rho'(x)=12x-6$, so $\rho'(x)=0$ if and only if $x=\frac{1}{2}$. since $\rho(0)=1$, $\rho\left(\frac{1}{2}\right)=-\frac{1}{2}$, and $\rho(1)=1$, we see that ρ attains a maximum value of 1 and a v,

$$s'((n,n+1))=\left[-\frac{1}{2},1\right]. \tag{6.4.11}$$

It follows from the preceding, in the same manner as the result in Example 5.1.7, that neither the function $\sigma(x)=s\left(\frac{1}{x}\right)$ nor the function $g(x)=s'\left(\frac{1}{x}\right)$ has a limit as x approaches 0.

Finally, define $\psi:\mathbb{R}\to\mathbb{R}$ by

$$\psi(x)=\begin{cases}x^2 s\left(\frac{1}{x}\right), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases} \tag{6.4.12}$$

For $x\neq 0$, we have

$$\psi'(x)=x^2 s'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+2xs\left(\frac{1}{x}\right)=-s'\left(\frac{1}{x}\right)+2xs\left(\frac{1}{x}\right). \tag{6.4.13}$$

At 0, we have

$$\begin{aligned}\psi'(0) &= \lim_{h\to 0}\frac{\psi(0+h)-\psi(0)}{h}\\ &= \lim_{h\to 0}\frac{h^2 s\left(\frac{1}{h}\right)}{h}\\ &= \lim_{h\to 0} hs\left(\frac{1}{h}\right)\\ &= 0,\end{aligned} \tag{6.4.14}$$

where the final limit follows from the squeeze theorem and the fact that s is bounded. Hence we see that ψ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$, but ψ' is not continuous since $\psi'(x)$ does not have a limit as x approaches 0. See Figure 6.4.1 for the

graphs of ψ and ψ'.

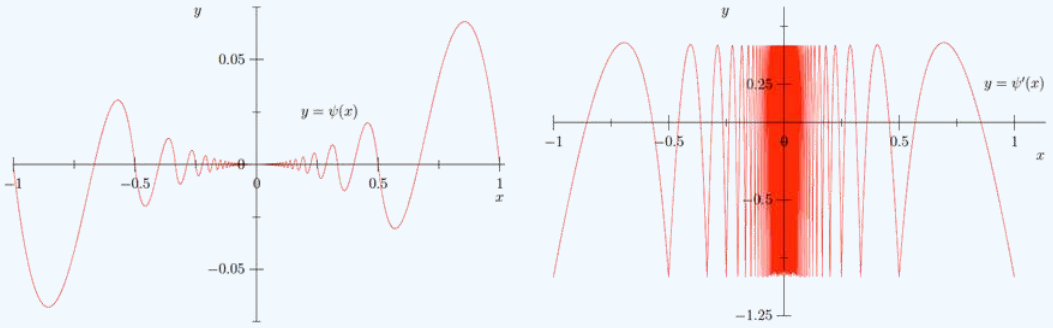

Figure 6.4.2: Graphs of $y = \psi(x)$ and $y = \psi'(x)$.

Exercise 6.4.3

Let s be as above and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^4 s\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases} \tag{6.4.15}$$

Show that g is differentiable on $\mathbb{R}$ and that g' is continuous on $\mathbb{R}$.

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6.5: L'Hopital's Rule

The following result is one case of l' l'Hópital's rule.

Theorem 6.5.1

Suppose $a, b \in \mathbb{R}$, f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lambda. \tag{6.5.1}$$

If $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda. \tag{6.5.2}$$

Proof

Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\lambda - \frac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < \lambda + \frac{\epsilon}{2} \tag{6.5.3}$$

whenever $x \in (a, a + \delta)$. Now, by the Generalized Mean Value Theorem, for any x and y with $a < x < y < a + \delta$, there exists a point $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(c)}{g'(c)}. \tag{6.5.4}$$

Hence

$$\lambda - \frac{\epsilon}{2} < \frac{f(y) - f(x)}{g(y) - g(x)} < \lambda + \frac{\epsilon}{2}. \tag{6.5.5}$$

Now

$$\lim_{x \to a^+} \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(y)}{g(y)} \tag{6.5.6}$$

and so we have

$$\lambda - \epsilon < \lambda - \frac{\epsilon}{2} \leq \frac{f(y)}{g(y)} \leq \lambda + \frac{\epsilon}{2} < \lambda + \epsilon \tag{6.5.7}$$

for any $y \in (a, a + \delta)$. Hence

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda. \tag{6.5.8}$$

Q.E.D.

Exercise 6.5.1

Use l'Hôpital's rule to compute

$$\lim_{x \to 0^+} \frac{\sqrt{1 + x} - 1}{x}.$$

Exercise 6.5.2

Suppose $a, b \in \mathbb{R}$, f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \lambda.$$

Show that if $\lim_{x \to b^-} f(x) = 0$ and $\lim_{x \to b^-} g(x) = 0$, then

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lambda.$$

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6.6: Taylor's Theorem

6.6.1 Derivatives of Higher Order

Definition

Suppose f is differentiable on an open interval I and f' is differentiable at $a \in I$. We call the derivative of f' at a the second derivative of f at a, which we denote $f''(a)$.

By continued differentiation, we may define the higher order derivatives f''', f'''', and so on. In general, for any integer $n, n \geq 0$, we let $f^{(n)}$ denote the nth derivative of f, where $f^{(0)}$ denotes f.

Exercise 6.6.1

Suppose $D \subset \mathbb{R}$, a is an interior point of D, $f : D \to \mathbb{R}$, and $f''(a)$ exists. Show that

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a). \tag{6.6.1}$$

Find an example to illustrate that this limit may exist even if $f''(a)$ does not exist.

For any open interval (a, b), where a and b are extended real numbers, we let $C^{(n)}(a, b)$, where $n \in \mathbb{Z}^+$, denote the set of all functions f with the property that each of $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ is defined and continuous on (a, b).

6.6.2 Taylor's Theorem

Theorem 6.6.1

(Taylor's Theorem).

Suppose $f \in C^{(n)}(a, b)$ and $f^{(n)}$ is differentiable on (a, b). Let $\alpha, \beta \in (a, b)$ with $\alpha \neq \beta$, and let

$$\begin{aligned} P(x) &= f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2}(x-\alpha)^2 + \cdots \\ &\quad + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k. \end{aligned} \tag{6.6.2}$$

Then there exists a point γ between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n+1)}(\gamma)}{(n+1)!}(\beta-\alpha)^{n+1}. \tag{6.6.3}$$

Proof

First note that $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, \ldots, n$. Let

$$M = \frac{f(\beta) - P(\beta)}{(\beta-\alpha)^{n+1}}. \tag{6.6.4}$$

Then

$$f(\beta) = P(\beta) + M(\beta-\alpha)^{n+1}. \tag{6.6.5}$$

We need to show that

$$M = \frac{f^{(n+1)}(\gamma)}{(n+1)!} \tag{6.6.6}$$

for some γ between α and β. Let

$$g(x) = f(x) - P(x) - M(x-\alpha)^{n+1}. \tag{6.6.7}$$

Then, for $k = 0, 1, \ldots, n$,

$$g^{(k)}(\alpha) = f^{(k)}(\alpha) - P^{(k)}(\alpha) = 0. \tag{6.6.8}$$

Now $g(\beta) = 0$, so, by Rolle's theorem, there exists γ_1 between α and β such that $g'(\gamma_1) = 0$. Using Rolle's theorem again, we see that there exists γ_2 between α and γ_1 such that $g''(\gamma_2) = 0$. Continuing for $n+1$ steps, we find γ_{n+1} between α and γ_n (and hence between α and β) such that $g^{(n+1)}(\gamma_{n+1}) = 0$. Hence

$$0 = g^{(n+1)}(\gamma_{n+1}) = f^{(n+1)}(\gamma_{n+1}) - (n+1)!M. \tag{6.6.9}$$

Letting $\gamma = \gamma_{n+1}$, we have

$$M = \frac{f^{(n+1)}(\gamma)}{(n+1)!}, \tag{6.6.10}$$

as required. Q.E.D.

We call the polynomial P in the statement of Taylor's theorem the Taylor polynomial of order n for f at α.

Example 6.6.1

Let $f(x) = \sqrt{x}$. Then the 4th order Taylor polynomial for f at 1 is

$$P(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4. \tag{6.6.11}$$

By Taylor's theorem, for any $x > 0$ there exists γ between 1 and x such that

$$\sqrt{x} = P(x) + \frac{105}{(32)(5!)\gamma^{\frac{9}{2}}}(x-1)^5 = P(x) + \frac{7}{256\gamma^{\frac{9}{2}}}(x-1)^5. \tag{6.6.12}$$

For example,

$$\begin{aligned} \sqrt{1.2} &= P(1.2) + \frac{7}{256\gamma^{\frac{9}{2}}}(1.2-1)^5 \\ &= P(1.2) + \frac{7}{256\gamma^{\frac{9}{2}}}(0.2)^5 = P(1.2) + \frac{7}{800000\gamma^{\frac{9}{2}}}, \end{aligned} \tag{6.6.13}$$

for some γ with $1 < \gamma < 1.2$. Hence $P(1.2)$ underestimates $\sqrt{1.2}$ by a value which is no larger than $\frac{7}{80000}$. Note that

$$P(1.2) = \frac{17527}{16000} = 1.0954375 \tag{6.6.14}$$

and

$$\frac{7}{800000} = 0.00000875. \tag{6.6.15}$$

So $\sqrt{1.2}$ lies between 1.0954375 and 1.09544625.

Exercise 6.6.2

Use the 5th order Taylor polynomial for $f(x) = \sqrt{x}$ at 1 to estimate $\sqrt{1.2}$. Is this an underestimate or an overestimate? Find an upper bound for the largest amount by which the estimate and $\sqrt{1.2}$ differ.

Exercise 6.6.3

Find the 3rd order Taylor polynomial for $f(x) = \sqrt[3]{1+x}$ at 0 and use it to estimate $\sqrt[3]{1.1}$. Is this an underestimate or an overestimate? Find an upper bound for the largest amount by which the estimate and $\sqrt[3]{1.1}$ differ.

Exercise 6.6.4

Suppose $f \in C^{(2)}(a,b)$. Use Taylor's theorem to show that

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = f''(c) \tag{6.6.16}$$

for any $c \in (a,b)$.

Exercise 6.6.5

Suppose $f \in C^{(1)}(a,b), c \in (a,b), f'(c) = 0$, and f'' exists on (a,b) and is continuous at c. Show that f has a local maximum at c if $f''(c) < 0$ and a local minimum at c if $f''(c) > 0$.

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CHAPTER OVERVIEW

7: Integrals

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7.2: Integrals
7.3: Integrability Conditions
7.4: Properties of Integrals
7.5: The Fundamental Theorem of Calculus
7.6: Taylor's Theorem Revisited
7.7: An Improper Integral

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7.1: Upper and Lower Integrals

Definition

Given a closed interval $[a,b] \subset \mathbb{R}$ with $a < b$, we call any finite subset of $[a,b]$ which includes both a and b a partition of $[a,b]$.

For convenience, whenever we consider a partition P of an interval $[a,b]$ we will index the elements in increasing order, starting with 0. That is, if $|P| = n+1$ and $P = \{x_0, x_1, \ldots, x_n\}$, then

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b. \tag{7.1.1}$$

Definition

Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is bounded. For $i = 1, 2, \ldots, n$, let

$$m_i = \inf \{f(x) : x_{i-1} \le x \le x_i\} \tag{7.1.2}$$

and

$$M_i = \sup \{f(x) : x_{i-1} \le x \le x_i\}. \tag{7.1.3}$$

We call

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \tag{7.1.4}$$

the lower sum of f determined by P and

$$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \tag{7.1.5}$$

the upper sum of f determined by P.

Definition

If P_1 and P_2 are both partitions of $[a,b]$ and $P_1 \subset P_2$, then we call P_2 a refinement of P_1.

Definition

If P_1 and P_2 are both partitions of $[a,b]$, then we call the partition $P = P_1 \cup P_2$ the common refinement of P_1 and P_2.

lemma 7.1.1

Suppose $P_1 = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, $s \in (a,b)$, $s \notin P_1$, and $f : [a,b] \to \mathbb{R}$ is bounded. If $P_2 = P_1 \cup \{s\}$, then $L(f,P_1) \le L(f,P_2)$ and $U(f,P_2) \le U(f,P_1)$.

Proof

Suppose $x_{i-1} < s < x_i$ and let

$$\begin{aligned} w_1 &= \inf \{f(x) : x_{i-1} \le x \le s\}, \\ W_1 &= \sup \{f(x) : x_{i-1} \le x \le s\}, \\ w_2 &= \inf \{f(x) : s \le x \le x_i\}, \\ W_2 &= \sup \{f(x) : s \le x \le x_i\}, \\ m_i &= \inf \{f(x) : x_{i-1} \le x \le x_i\}, \end{aligned} \tag{7.1.6}$$

and

$$M_i = \sup \{f(x) : x_{i-1} \le x \le x_i\}. \tag{7.1.7}$$

Then $w_1 \geq m_i, w_2 \geq m_i, W_1 \leq M_i$, and $W_2 \leq M_i$. Hence

$$\begin{aligned} L(f,P_2) - L(f,P_1) &= w_1(s - x_{i-1}) + w_2(x_i - s) - m_i(x_i - x_{i-1}) \\ &= w_1(s - x_{i-1}) + w_2(x_i - s) - m_i(s - x_{i-1}) \\ &\quad -m_i(x_i - s) \\ &= (w_1 - m_i)(s - x_{i-1}) + (w_2 - m_i)(x_i - s) \\ &\geq 0 \end{aligned} \tag{7.1.8}$$

and

$$\begin{aligned} U(f,P_1) - U(f,P_2) =& M_i(x_i - x_{i-1}) - W_1(s - x_{i-1}) - W_2(x_i - s) \\ =& M_i(s - x_{i-1}) + M_i(x_i - s) - W_1(s - x_{i-1}) \\ &- W_2(x_i - s) \\ =& (M_i - W_1)(s - x_{i-1}) + (M_i - W_2)(x_i - s) \\ \geq& 0. \end{aligned} \tag{7.1.9}$$

Thus $L(f,P_1) \leq L(f,P_2)$ and $U(f,P_2) \leq U(f,P_1)$. Q.E.D.

Proposition 7.1.2

Suppose P_1 and P_2 are partitions of $[a,b]$, with P_2 a refinement of P_1. If $f : [a,b] \to \mathbb{R}$ is bounded, then $L(f,P_1) \leq L(f,P_2)$ and $U(f,P_2) \leq U(f,P_1)$.

Proof

The proposition follows immediately from repeated use of the previous lemma. Q.E.D.

Proposition 7.1.3

Suppose P_1 and P_2 are partitions of $[a,b]$. If $f : [a,b] \to \mathbb{R}$ is bounded, then $L(f,P_1) \leq U(f,P_2)$.

Proof

The result follows immediately from the definitions if $P_1 = P_2$. Otherwise, let P be the common refinement of P_1 and P_2. Then

$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2). \tag{7.1.10}$$

Q.E.D.

Definition

Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is bounded. We call

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \tag{7.1.11}$$

the lower integral of f over $[a,b]$ and

$$\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} \tag{7.1.12}$$

the upper integral of f over $[a,b]$.

Note that both the lower integral and the upper integral are finite real numbers since the lower sums are all bounded above by any upper sum and the upper sums are all bounded below by any lower sum.

Proposition 7.1.4

Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is bounded. Then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f. \tag{7.1.13}$$

Proof

Let P be a partition of $[a, b]$. Then for any partition Q of $[a, b]$, we have $L(f, Q) \leq U(f, P)$. Hence $U(f, P)$ is an upper bound for any lower sum, and so

$$\underline{\int_a^b} f \leq U(f, P). \tag{7.1.14}$$

But this shows that the lower integral is a lower bound for any upper sum. Hence

$$\int_a^b f \leq \int_a^b f. \tag{7.1.15}$$

Q.E.D.

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7.2: Integrals

Definition

Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is bounded. We say f is

$$\underline{\int_a^b} f = \overline{\int_a^b} f. \tag{7.2.1}$$

If f is integrable, we call the common value of the upper and lower integrals the integral of f over $[a, b]$, denoted

$$\int_a^b f. \tag{7.2.2}$$

That is, if f is integrable on $[a, b]$,

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f. \tag{7.2.3}$$

Example 7.2.1

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases} \tag{7.2.4}$$

For any partition $P = \{x_0, x_1, \dots, x_n\}$, we have

$$L(f, P) = \sum_{i=1}^{n} 0\,(x_i - x_{i-1}) = 0 \tag{7.2.5}$$

and

$$U(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = 1. \tag{7.2.6}$$

Thus

$$\underline{\int_0^1} f = 0 \tag{7.2.7}$$

and

$$\overline{\int_0^1} f = 1. \tag{7.2.8}$$

Hence f is not integrable on $[0, 1]$.

Example 7.2.2

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational}, \end{cases} \tag{7.2.9}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that f is integrable on $[0, 1]$ and

$$\int_0^1 f = 0. \tag{7.2.10}$$

Exercise 7.2.3

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$ and, for $n \in \mathbb{Z}^+$, let $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[0,1]$ with

$$x_i = \frac{i}{n}, i = 0, 1, \ldots, n. \tag{7.2.11}$$

Show that

$$U(f,P) - L(f,P) = \frac{1}{n}, \tag{7.2.12}$$

and hence conclude that f is integrable on $[0,1]$. Show that

$$\int_0^1 f = \frac{1}{2}. \tag{7.2.13}$$

Exercise 7.2.4

Define $f : [1,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases} \tag{7.2.14}$$

Show that f is not integrable on $[1,2]$.

Exercise 7.2.5

Suppose f is integrable on $[a,b]$, and, for some real number m and $M, m \le f(x) \le M$ for all $x \in [a,b]$. Show that

$$m(b-a) \le \int_a^b f \le M(b-a). \tag{7.2.15}$$

7.2.1 Notation and Terminology

The definition of the integral described in this section is due to Darboux. One may show it to be equivalent to the integral defined by Riemann. Hence functions that are integrable in the sense of this discussion are referred to as Riemann integrable functions and we call the integral the Riemann integral. This is in distinction to the Lebesgue integral, part of a more general theory of integration.

We sometimes refer to this integral as the definite integral, as opposed to an indefinite integral, the latter being a name given to an antiderivative (a function whose derivative is equal to a given function).

If f is integrable on $[a,b]$, then we will also denote

$$\int_a^b f \tag{7.2.16}$$

by

$$\int_a^b f(x)dx. \tag{7.2.17}$$

The variable x in the latter is a "dummy" variable; we may just as well write

$$\int_a^b f(t)dt \tag{7.2.18}$$

or

$$\int_a^b f(s)ds. \tag{7.2.19}$$

For example, if $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = x^2$, then

$$\int_0^1 f = \int_0^1 x^2 dx = \int_0^1 t^2 dt. \tag{7.2.20}$$

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7.3: Integrability Conditions

Proposition 7.3.1

If $a < b$ and $f : [a, b] \to \mathbb{R}$ is monotonic, then f is integrable on $[a, b]$.

Proof

Suppose f is nondecreasing. Given $\epsilon > 0$, let $n \in Z^+$ be large enough that

$$\frac{(f(b) - f(a))(b - a)}{n} < \epsilon. \tag{7.3.1}$$

For $i = 0, 1, \ldots, n$, let

$$x_i = a + \frac{(b - a)i}{n}. \tag{7.3.2}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \frac{b - a}{n} \\ &= \frac{b - a}{n} ((f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \cdots \\ &\quad + (f(x_{n-1}) - f(x_{n-2})) + (f(x_n) - f(x_{n-1}))) \\ &= \frac{b - a}{n}(f(b) - f(a)) \\ &< \epsilon. \end{aligned} \tag{7.3.3}$$

Hence f is integrable on $[a, b]$. Q.E.D.

Example 7.3.1

Let $\varphi : \mathbb{Q} \cap [0, 1] \to \mathbb{Z}^+$ be a one-to-one correspondence. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{q \in \mathbb{Q} \cap [0,1] \\ q \le x}} \frac{1}{2^{\varphi(q)}}. \tag{7.3.4}$$

Then f is increasing on $[0, 1]$, and hence integrable on $[0, 1]$.

Proposition 7.3.2

If $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on $[a, b]$.

Proof

Given $\epsilon > 0$, let

$$\gamma = \frac{\epsilon}{b - a}. \tag{7.3.5}$$

Since f is uniformly continuous on $[a, b]$, we may choose $\delta > 0$ such that

$$|f(x) - f(y)| < \gamma \tag{7.3.6}$$

whenever $|x - y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with

$$\sup \{|x_i - x_{i-1}| : i = 1, 2, \ldots, n\} < \delta. \tag{7.3.7}$$

If, for $i = 1, 2, \ldots, n$,

$$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\} \tag{7.3.8}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}, \tag{7.3.9}$$

then $M_i - m_i < \gamma$. Hence

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \gamma(b-a) \\ &= \epsilon. \end{aligned} \tag{7.3.10}$$

Thus f is integrable on $[a,b]$. Q.E.D.

Exercise 7.3.1

Suppose $a < b$, $f : [a,b] \to \mathbb{R}$ is bounded, and $c \in [a,b]$. Show that if f is continuous on $[a,b]\backslash\{c\}$, then f is integrable on $[a,b]$.

Exercise 7.3.2

Suppose $a < b$ and f is continuous on $[a,b]$ with $f(x) \ge 0$ for all $x \in [a,b]$. Show that if

$$\int_a^b f = 0, \tag{7.3.11}$$

then $f(x) = 0$ for all $x \in [a,b]$.

Exercise 7.3.3

Suppose $a < b$ and f is continuous on $[a,b]$. For $i = 0, 1, \ldots, n$, $n \in \mathbb{Z}^+$, let

$$x_i = a + \frac{(b-a)i}{n} \tag{7.3.12}$$

and, for $i = 1, 2, \ldots, n$, let $c_i \in [x_{i-1}, x_i]$. Show that

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(c_i). \tag{7.3.13}$$

In the notation of Exercise 7.3.3, we call the approximation

$$\int_a^b f \approx \frac{b-a}{n} \sum_{i=1}^{n} f(c_i) \tag{7.3.14}$$

a right-hand rule approximation if $c_i = x_i$, a left-hand rule approximation if $c_i = x_{i-1}$, and a midpoint rule approximation if

$$c_i = \frac{x_{i-1} + x_i}{2}. \tag{7.3.15}$$

These are basic ingredients in creating numerical approximations to integrals.

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7.4: Properties of Integrals

Proposition 7.4.1

If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, then

$$\sup\{f(x) + g(x) : x \in D\} \leq \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\} \tag{7.4.1}$$

and

$$\inf\{f(x) + g(x) : x \in D\} \geq \inf\{f(x) : x \in D\} + \inf\{g(x) : x \in D\} \tag{7.4.2}$$

Exercise 7.4.1

Prove the previous proposition.

Exercise 7.4.2

Find examples for which the inequalities in the previous proposition are strict.

Proposition 7.4.2

Suppose f and g are both integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g. \tag{7.4.3}$$

Proof

Given $\epsilon > 0$, let P_1 and P_2 be partitions of $[a, b]$ with

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \tag{7.4.4}$$

and

$$U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}. \tag{7.4.5}$$

Let $P = P_1 \cup P_2$. By the previous proposition,

$$U(f + g, P) \leq U(f, P) + U(g, P) \tag{7.4.6}$$

and

$$L(f + g, P) \geq L(f, P) + L(g, P). \tag{7.4.7}$$

Hence

$$\begin{aligned} U(f + g, P) - L(f + g, P) &\leq (U(f, P) + U(g, P)) - (L(f, P) + L(g, P)) \\ &= (U(f, P) - L(f, P)) + (U(g, P) - L(g, P)) \\ &\leq (U(f, P_1) - L(f, P_1)) + (U(g, P_2) - L(g, 2P)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{7.4.8}$$

Hence $f + g$ is integrable on $[a, b]$.

Moreover,

$$\begin{aligned} \int_a^b (f + g) &\leq U(f + g, P) \\ &\leq U(f, P) + U(g, P) \\ &\leq \left(\int_a^b f + \frac{\epsilon}{2}\right) + \left(\int_a^b g + \frac{\epsilon}{2}\right) \end{aligned} \tag{7.4.9}$$

$$= \int_a^b f + \int_a^b g + \epsilon$$

and

$$\begin{aligned} \int_a^b (f+g) &\geq L(f+g, P) \\ &\geq L(f,P) + L(g,P) \\ &\geq \left(\int_a^b f - \frac{\epsilon}{2} \right) + \left(\int_a^b g - \frac{\epsilon}{2} \right) \\ &= \int_a^b f + \int_a^b g - \epsilon. \end{aligned} \tag{7.4.10}$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g. \tag{7.4.11}$$

Q.E.D.

Exercise 7.4.3

Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both bounded. Show that

$$\overline{\int_a^b} (f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g. \tag{7.4.12}$$

Find an example for which the inequality is strict.

Exercise 7.4.4

Find an example to show that $f+g$ may be integrable on $[a,b]$ even though neither f nor g is integrable on $[a,b]$.

Proposition 7.4.3

If f is integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, then αf is integrable on $[a,b]$ and

$$\int_a^b \alpha f = \alpha \int_a^b f. \tag{7.4.13}$$

Exercise 7.4.5

Prove the previous proposition.

Theorem 7.4.4

Suppose $a < b, f : [a,b] \to \mathbb{R}$ is bounded, and $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on both $[a,c]$ and $[c,b]$.

Proof

Suppose f is integrable on $[a,b]$. Given $\epsilon > 0$, let Q be a partition of $[a,b]$ such that

$$U(f,Q) - L(f,Q) < \epsilon. \tag{7.4.14}$$

Let $P = Q \cup \{c\}, P_1 = P \cap [a,c]$, and $P_2 = P \cap [c,b]$. Then

$$\begin{aligned} (U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) &= (U(f,P_1) + U(f,P_2)) \\ &\quad - (L(f,P_1) + L(f,P_2)) \\ &= U(f,P) - L(f,P) \end{aligned} \tag{7.4.15}$$

$$\begin{aligned} &\leq U(f,Q) - L(f,Q) \\ &< \epsilon. \end{aligned}$$

Thus we must have both

$$U(f,P_1) - L(f,P_1) < \epsilon \tag{7.4.16}$$

and

$$U(f,P_2) - L(f,P_2) < \epsilon. \tag{7.4.17}$$

Hence f is integrable on both $[a,c]$ and $[c,b]$.

Now suppose f is integrable on both $[a,c]$ and $[c,b]$. Given $\epsilon > 0$, let P_1 and P_2 be partitions of $[a,c]$ and $[c,b]$, respectively, such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2} \tag{7.4.18}$$

and

$$U(f,P_2) - L(f,P_2) < \frac{\epsilon}{2}. \tag{7.4.19}$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a,b]$ and

$$\begin{aligned} U(f,P) - L(f,P) &= (U(f,P_1) + U(f,P_2)) - (L(f,P_1) + L(f,P_2)) \\ &= (U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{7.4.20}$$

Thus f is integrable on $[a,b]$. Q.E.D.

Proposition 7.4.5

Suppose f is integrable on $[a,b]$ and $c \in (a,b)$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{7.4.21}$$

Proof

If P and Q are partitions of $[a,c]$ and $[c,b]$, respectively, then

$$U(f,P) + U(f,Q) = U(f,P \cup Q) \geq \int_a^b f. \tag{7.4.22}$$

Thus

$$U(f,P) \geq \int_a^b f - U(f,Q), \tag{7.4.23}$$

so

$$\int_a^c f = \overline{\int_a^c} f \geq \int_a^b f - U(f,Q). \tag{7.4.24}$$

Hence

$$U(f,Q) \geq \int_a^b f - \int_a^c f, \tag{7.4.25}$$

so

$$\int_c^b f = \overline{\int_c^b} f \geq \int_a^b f - \int_a^c f. \tag{7.4.26}$$

Thus

$$\int_a^c f + \int_c^b f \geq \int_a^b f. \tag{7.4.27}$$

Similarly, if P and Q are partitions of $[a,c]$ and $[c,b]$, respectively, then

$$L(f,P) + L(f,Q) = L(f, P \cup Q) \leq \int_a^b f. \tag{7.4.28}$$

Thus

$$L(f,P) \leq \int_a^b f - L(f,Q), \tag{7.4.29}$$

so

$$\int_a^c f = \underline{\int_a^c} f \leq \int_a^b f - L(f,Q). \tag{7.4.30}$$

Hence

$$L(f,Q) \leq \int_a^b f - \int_a^c f, \tag{7.4.31}$$

so

$$\int_c^b f = \underline{\int_c^b} f \leq \int_a^b f - \int_a^c f. \tag{7.4.32}$$

Thus

$$\int_a^c f + \int_c^b f \leq \int_a^b f. \tag{7.4.33}$$

Hence

$$\int_a^c f + \int_c^b f = \int_a^b f. \tag{7.4.34}$$

Q.E.D.

? Exercise 7.4.6

Suppose $f : [a,b] \to \mathbb{R}$ is bounded and B is a finite subset of (a,b). Show that if f is continuous on $[a,b] \backslash B$, then f is integrable on $[a,b]$.

Proposition 7.4.6

If f is integrable on $[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$, then

$$\int_a^b f \geq 0. \tag{7.4.35}$$

Proof

The result follows from the fact that $L(f,P) \geq 0$ for any partition P of $[a,b]$. Q.E.D.

Proposition 7.4.7

Suppose f and g are both integrable on $[a,b]$. If, for all $x \in [a,b]$, $f(x) \leq g(x)$, then

$$\int_a^b f \leq \int_a^b g. \tag{7.4.36}$$

Proof

Since $g(x) - f(x) \geq 0$ for all $x \in [a,b]$, we have, using Propositions 7.4.2, 7.4.3, and 7.4.6,

$$\int_a^b g - \int_a^b f = \int_a^b (g-f) \geq 0. \tag{7.4.37}$$

Q.E.D.

Proposition 7.4.8

Suppose f is integrable on $[a,b]$, $m \in \mathbb{R}$, $M \in \mathbb{R}$, and $m \leq f(x) \leq M$ for all $x \in [a,b]$. Then

$$m(b-a) \leq \int_a^b f \leq M(b-a). \tag{7.4.38}$$

Proof

It follows from the previous proposition that

$$m(b-a) = \int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx = M(b-a). \tag{7.4.39}$$

Q.E.D.

Exercise 7.4.7

Show that

$$1 \leq \int_{-1}^{1} \frac{1}{1+x^2} dx \leq 2. \tag{7.4.40}$$

Exercise 7.4.8

Suppose f is continuous on $[0,1]$, differentiable on $(0,1)$, $f(0) = 0$, and $|f'(x)| \leq 1$ for all $x \in (0,1)$. Show that

$$-\frac{1}{2} \leq \int_0^1 f \leq \frac{1}{2}. \tag{7.4.41}$$

Exercise 7.4.9

Suppose f is integrable on $[a,b]$ and define $F : (a,b) \to \mathbb{R}$ by

$$F(x) = \int_a^x f. \tag{7.4.42}$$

Show that there exists $\alpha \in \mathbb{R}$ such that for any $x, y \in (a,b)$ with $x < y$,

$$|F(y) - F(x)| \leq \alpha(y-x). \tag{7.4.43}$$

Proposition 7.4.9

Suppose g is integrable on $[a,b]$, $g([a,b]) \subset [c,d]$, and $f : [c,d] \to \mathbb{R}$ is continuous. If $h = f \circ g$, then h is integrable on $[a,b]$.

Proof

Let $\epsilon > 0$ be given. Let

$$K > \sup\{f(x) : x \in [c,d]\} - \inf\{f(x) : x \in [c,d]\} \tag{7.4.44}$$

and choose $\delta > 0$ so that $\delta < \epsilon$ and

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \tag{7.4.45}$$

whenever $|x - y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that

$$U(g,P) - L(g,P) < \frac{\delta^2}{2K}. \tag{7.4.46}$$

For $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{g(x) : x_{i-1} \le x \le x_i\}, \tag{7.4.47}$$

$$M_i = \sup\{g(x) : x_{i-1} \le x \le x_i\}, \tag{7.4.48}$$

$$w_i = \inf\{h(x) : x_{i-1} \le x \le x_i\}, \tag{7.4.49}$$

and

$$W_i = \sup\{h(x) : x_{i-1} \le x \le x_i\}. \tag{7.4.50}$$

Finally, let

$$I = \{i : i \in \mathbb{Z}, 1 \le i \le n, M_i - m_i < \delta\} \tag{7.4.51}$$

and

$$J = \{i : i \in \mathbb{Z}, 1 \le i \le n, M_i - m_i \ge \delta\}. \tag{7.4.52}$$

Note that

$$\begin{aligned} \delta \sum_{i \in J} (x_i - x_{i-1}) &\le \sum_{i \in J} (M_i - m_i)(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\delta^2}{2K}, \end{aligned} \tag{7.4.53}$$

from which it follows that

$$\sum_{i \in J} (x_i - x_{i-1}) < \frac{\delta}{2K}. \tag{7.4.54}$$

Then

$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i \in I} (W_i - w_i)(x_i - x_{i-1}) + \sum_{i \in J} (W_i - w_i)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2(b-a)} \sum_{i \in I} (x_i - x_{i-1}) + K \sum_{i \in J} (x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} + \frac{\delta}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{7.4.55}$$

Thus h is integrable on $[a,b]$. Q.E.D.

Proposition 7.4.10

Suppose f and g are both integrable on $[a,b]$. Then fg is integrable on $[a,b]$.

Proof

Since f and g are both integrable, both $f+g$ and $f-g$ are integrable. Hence, by the previous proposition, both $(f+g)^2$ and $(f-g)^2$ are integrable. Thus

$$\frac{1}{4}\left((f+g)^2-(f-g)^2\right)\Big) = fg \tag{7.4.56}$$

is integrable on $[a,b]$. Q.E.D.

Proposition 7.4.11

Suppose f is integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f\right| \le \int_a^b |f|. \tag{7.4.57}$$

Proof

The integrability of $|f|$ follows from the integrability of f, the continuity of $g(x)=|x|$, and Proposition 7.4.9. For the inequality, note that

$$-|f(x)| \le f(x) \le |f(x)| \tag{7.4.58}$$

for all $x \in [a,b]$. Hence

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|, \tag{7.4.59}$$

from which the result follows. Q.E.D.

Exercise 7.4.10

Either prove the following statement or show it is false by finding a counterexample: If $f:[0,1]\to\mathbb{R}$ is bounded and f^2 is integrable on $[0,1]$, then f is integrable on $[0,1]$.

7.4.1 Extended definitions

Definition

If f integrable on $[a,b]$, then we define

$$\int_b^a f = -\int_a^b f. \tag{7.4.60}$$

Moreover, if f is a function defined at a point $a \in \mathbb{R}$, we define

$$\int_a^a f = 0. \tag{7.4.61}$$

Definition

Suppose f is integrable on a closed interval containing the

points a, b, and c. Show that

$$\int^b f = \int^c f + \int^b f. \tag{7.4.62}$$

$\int_a^a = \int_a^a = \int_c^c$

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7.5: The Fundamental Theorem of Calculus

Theorem 7.5.1

(Fundamental Theorem of Calculus)

Suppose f is integrable on $[a,b]$. If F is continuous on $[a,b]$ and differentiable on (a,b) with $F'(x)=f(x)$ for all $x\in(a,b)$, then

$$\int_a^b f = F(b)-F(a). \tag{7.5.1}$$

Proof

Given $\epsilon>0$, let $P=\{x_0,x_1,\ldots,x_n\}$ be a partition of $[a,b]$ for which

$$U(f,P)-L(f,P)<\epsilon. \tag{7.5.2}$$

For $i=1,2,\ldots,n$, let $t_i\in(x_{i-1},x_i)$ be points for which

$$F(x_i)-F(x_{i-1})=f(t_i)(x_i-x_{i-1}). \tag{7.5.3}$$

Then

$$\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})=\sum_{i=1}^{n}(F(x_i)-F(x_{i-1}))=F(b)-F(a). \tag{7.5.4}$$

But

$$L(f,P)\le\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})\le U(f,P), \tag{7.5.5}$$

so

$$\left|F(b)-F(a)-\int_a^b f\right|<\epsilon. \tag{7.5.6}$$

Since ϵ was arbitrary, we conclude that

$$\int_a^b f = F(b)-F(a). \tag{7.5.7}$$

Q.E.D.

Proposition 7.5.2

(Integration by parts)

Suppose f and g are integrable

on $[a,b]$. If F and G are continuous on $[a,b]$ and differentiable on (a,b) with $F'(x)=f(x)$ and $G'(x)=g(x)$ for all $x\in(a,b)$, then

$$\int_a^b F(x)g(x)dx = F(b)G(b)-F(a)G(a)-\int_a^b f(x)G(x)dx. \tag{7.5.8}$$

Proof

By the Fundamental Theorem of Calculus,

$$\int_a^b (F(x)g(x)+f(x)G(x))dx = F(b)G(b)-F(a)G(a). \tag{7.5.9}$$

Q.E.D.

7.5.1 The other Fundamental Theorem of Calculus

Proposition 7.5.3

Suppose f is integrable on $[a,b]$ and $F:[a,b]\to\mathbb{R}$ is defined by

$$F(x)=\int_a^x f(t)dt. \tag{7.5.10}$$

Then F is uniformly continuous on $[a,b]$.

Proof

Let $\epsilon>0$ be given and let $M>0$ be such that $|f(x)|\le M$ for all $x\in[a,b]$. Then for any $x,y\in[a,b]$ with $x<y$ and $y-x<\frac{\epsilon}{M}$,

$$|F(y)-F(x)|=\left|\int_x^y f(t)dt\right|\le M(y-x)<\epsilon. \tag{7.5.11}$$

Hence F is uniformly continuous on $[a,b]$. Q.E.D.

The following theorem is often considered to be part of the Fundamental Theorem of Calculus.

Theorem 7.5.4

Suppose f is integrable on $[a,b]$ and continuous at $u\in(a,b)$. If $F:[a,b]\to\mathbb{R}$ is defined by

$$F(x)=\int_a^x f(t)dt, \tag{7.5.12}$$

then F is differentiable at u and $F'(u)=f(u)$.

Proof

Let $\epsilon>0$ be given and choose $\delta>0$ such that $|f(x)-f(u)|<\epsilon$ whenever $|x-u|<\delta$. Then if $0<h<\delta$, we have

$$\begin{aligned}\left|\frac{F(u+h)-F(u)}{h}-f(u)\right|&=\left|\frac{1}{h}\int_u^{u+h}f(t)dt-f(u)\right|\\&=\left|\frac{1}{h}\int_u^{u+h}(f(t)-f(u))dt\right|\\&<\epsilon.\end{aligned} \tag{7.5.13}$$

If $-\delta<h<0$, then

$$\begin{aligned}\left|\frac{F(u+h)-F(u)}{h}-f(u)\right|&=\left|-\frac{1}{h}\int_{u+h}^{u}f(t)dt-f(u)\right|\\&=\left|\frac{1}{h}\int_{u+h}^{u}f(t)dt+f(u)\right|\\&=\left|\frac{1}{h}\int_{u+h}^{u}f(t)dt-\frac{1}{h}\int_{u+h}^{u}f(u)dt\right|\\&=\left|\frac{1}{h}\int_{u+h}^{u}(f(t)-f(u))dt\right|\\&<\epsilon.\end{aligned} \tag{7.5.14}$$

Hence

$$F'(u)=\lim_{h\to0}\frac{F(u+h)-F(u)}{h}=f(u). \tag{7.5.15}$$

Q.E.D.

Proposition 7.5.5

If $a < b$ and f is continuous on $[a, b]$, then there exists a function $F : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ with $F'(x) = f(x)$ for all $x \in (a, b)$.

Proof

Let

$$F(x) = \int_a^x f(t)dt. \tag{7.5.16}$$

Q.E.D.

Example 7.5.1

If

$$g(x) = \int_0^x \sqrt{1 + t^4}dt, \tag{7.5.17}$$

then $g'(x) = \sqrt{1 + x^4}$.

Proposition 7.5.6

(Integration by substitution)

Suppose I is an open interval, $\varphi : I \to \mathbb{R}, a < b, [a, b] \subset I$, and φ' is continuous on $[a, b]$. If $f : \varphi([a, b]) \to \mathbb{R}$ is continuous, then

$$\int_{\varphi(a)}^{\varphi(b)} f(u)du = \int_a^b f(\varphi(x))\varphi'(x)dx. \tag{7.5.18}$$

Proof

If m and M are the minimum and maximum values, respectively, of φ on $[a, b]$, then $\varphi([a, b]) = [m, M]$. If $m = M$, then $\varphi(x) = m$ for all $x \in [a, b]$, and both sides of (7.5.17) are 0. So we may assume $m < M$. Let F be a function which is continuous on $[m, M]$ with $F'(u) = f(u)$ for every $u \in (m, M)$. Let $g = F \circ \varphi$. Then

$$g'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x). \tag{7.5.19}$$

So if $\varphi(a) \le \varphi(b)$,

$$\begin{aligned} \int_a^b f(\varphi(x))\varphi'(x)dx &= g(b) - g(a) \\ &= F(\varphi(b)) - F(\varphi(a)) \\ &= \int_{\varphi(a)}^{\varphi(b)} f(u)du. \end{aligned} \tag{7.5.20}$$

If $\varphi(a) > \varphi(b)$, then

$$\begin{aligned} \int_a^b f(\varphi(x))\varphi'(x)dx &= g(b) - g(a) \\ &= F(\varphi(b)) - F(\varphi(a)) \\ &= -(F(\varphi(a)) - F(\varphi(b))) \\ &= -\int_{\varphi(b)}^{\varphi(a)} f(u)du \\ &= \int_{\varphi(a)}^{\varphi(b)} f(u)du. \end{aligned} \tag{7.5.21}$$

Q.E.D.

Exercise 7.5.1

Evaluate

$$\int_0^1 u\sqrt{u+1}du \tag{7.5.22}$$

using (a) integration by parts and (b) substitution.

Exercise 7.5.2

Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and periodic with period 1(that is, $\varphi(x+1) = \varphi(x)$ for every $x \in \mathbb{R}$). Show that for any continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^1 f(\varphi(x))\varphi'(x)dx = 0. \tag{7.5.23}$$

Theorem 7.5.7

(Integral Mean Value Theorem)

If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f = f(c)(b-a). \tag{7.5.24}$$

Exercise 7.5.3

Prove the Integral Mean Value Theorem.

Theorem 7.5.8

(Generalized Integral Mean Value Theorem)

If f and g are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b fg = f(c)\int_a^b g. \tag{7.5.25}$$

Exercise 7.5.4

Prove the Generalized Integral Mean Value Theorem.

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7.6: Taylor's Theorem Revisited

The following is a version of Taylor's Theorem with an alternative form of the remainder term.

Theorem 7.6.1

(Taylor's Theorem)

Suppose $f \in C^{(n+1)}(a,b), \alpha \in (a,b)$, and

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k. \tag{7.6.1}$$

Then, for any $x \in (a,b)$,

$$f(x) = P_n(x) + \int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt. \tag{7.6.2}$$

Proof

By the Fundamental Theorem of Calculus, we have

$$\int_\alpha^x f'(t)dt = f(x) - f(\alpha), \tag{7.6.3}$$

which implies that

$$f(x) = f(\alpha) + \int_\alpha^x f'(t)dt. \tag{7.6.4}$$

Hence the theorem holds for $n = 0$. Suppose the result holds for $n = k-1$, that is,

$$f(x) = P_{k-1}(x) + \int_\alpha^x \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} dt. \tag{7.6.5}$$

Let

$$F(t) = f^{(k)}(t), \tag{7.6.6}$$

$$g(t) = \frac{(x-t)^{k-1}}{(k-1)!}, \tag{7.6.7}$$

and

$$G(t) = -\frac{(x-t)^k}{k!}. \tag{7.6.8}$$

Then

$$\begin{aligned} \int_\alpha^x \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}dt &= \int_\alpha^x F(t)g(t)dt \\ &= F(x)G(x) - F(\alpha)G(\alpha) - \int_\alpha^x F'(t)G(t)dt \\ &= \frac{f^{(k)}(\alpha)(x-\alpha)^k}{k!} + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt, \end{aligned} \tag{7.6.9}$$

Hence

$$f(x) = P_k(x) + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt, \tag{7.6.10}$$

and so the theorem holds for $n = k$. Q.E.D.

Exercise 7.6.1

(Cauchy form of the remainder)

Under the conditions of Taylor's Theorem as just stated, show that

$$\int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n+1)}(\gamma)}{n!}(x-\gamma)^n(x-\alpha) \tag{7.6.11}$$

for some γ between α and x.

Exercise 7.6.2

(Lagrange form of the remainder)

Under the conditions of Taylor's Theorem as just stated, show that

$$\int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n+1)}(\gamma)}{(n+1)!}(x-\alpha)^{n+1} \tag{7.6.12}$$

for some γ between α and x. Note that this is the form of the remainder in Theorem 6.6.1, although under slightly more restrictive assumptions.

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7.7: An Improper Integral

Definition

If f integrable on $[a,b]$ for all $b>a$ and

$$\lim_{b\to+\infty}\int_a^b f(x)dx \tag{7.7.1}$$

exists, then we define

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx. \tag{7.7.2}$$

If f is integrable on $[a,b]$ for all $a<b$ and

$$\lim_{a\to-\infty}\int_a^b f(x)dx \tag{7.7.3}$$

exists, then we define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx. \tag{7.7.4}$$

Both of these integrals are examples of improper integrals.

Proposition 7.7.1

Suppose f is continuous on $[a,\infty)$ and $f(x)\geq 0$ for all $x\geq a$. If there exists $g:[a,+\infty)\to\mathbb{R}$ for which

$$\int_a^{+\infty} g(x)dx \tag{7.7.5}$$

exists and $g(x)\geq f(x)$ for all $x\geq a$, then

$$\int_a^{+\infty} f(x)dx \tag{7.7.6}$$

exists.

Exercise 7.7.1

Prove the preceding proposition.

Example 7.7.1

Suppose

$$f(x)=\frac{1}{1+x^2} \tag{7.7.7}$$

and

$$g(x)=\begin{cases} 1, & \text{if } 0\leq x<1,\\ \frac{1}{x^2}, & \text{if } x\geq 1.\end{cases} \tag{7.7.8}$$

Then, for $b>1$

$$\int_0^b g(x)dx=\int_0^1 dx+\int_1^b \frac{1}{x^2}dx=1+1-\frac{1}{b}=2-\frac{1}{b}, \tag{7.7.9}$$

so

$$\int_0^{+\infty} g(x)dx = \lim_{b\to\infty}\left(2-\frac{1}{b}\right) = 2. \tag{7.7.10}$$

Since $0 < f(x) \leq g(x)$ for all $x \geq 0$, it follows that

$$\int_0^{+\infty} \frac{1}{1+x^2}dx \tag{7.7.11}$$

exists, and, moreover,

$$\int_0^{+\infty} \frac{1}{1+x^2}dx < 2. \tag{7.7.12}$$

Also, the substitution $u = -x$ shows that

$$\int_{-\infty}^0 \frac{1}{1+x^2}dx = -\int_{+\infty}^0 \frac{1}{1+u^2}du = \int_0^{+\infty} \frac{1}{1+u^2}du. \tag{7.7.13}$$

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CHAPTER OVERVIEW

8: More Functions

8.1: The Arctangent Function
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8.3: The Sine and Cosine Functions
8.4: The Logarithm Functions
8.5: The Exponential Function

Thumbnail: Periodic functions. (CC BY-NC-SA; OpenStax)

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8.1: The Arctangent Function

Definition

For any $x \in \mathbb{R}$, we call

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt \tag{8.1.1}$$

the arctangent of x.

Proposition 8.1.1

The arctangent function is differentiable at every $x \in \mathbb{R}$.

Moreover, if $f(x) = \arctan(x)$, then

$$f'(x) = \frac{1}{1+x^2}. \tag{8.1.2}$$

Proof

The result follows immediately from Theorem 7.5.4. Q.E.D.

Proposition 8.1.2

The arctangent is increasing on $\mathbb{R}$.

Proof

The result follows immediately from the previous proposition and the fact that

$$\frac{1}{1+x^2} > 0 \tag{8.1.3}$$

for every $x \in \mathbb{R}$. Q.E.D.

Definition

$\pi = 2\lim_{x\to+\infty}\arctan(x) = 2\int_0^{+\infty}\frac{1}{1+t^2}dt$

Note that $0 < \pi < 4$ by Example 7.7.1.

The following proposition says that the arctangent function is an odd function.

Theorem 8.1.3

For any $x \in \mathbb{R}$, arctan $(x) = -\arctan(-x)$.

Proof

Using the substitution $t = -u$, we have

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt = -\int_0^{-x} \frac{1}{1+u^2}\,du = -\arctan(-x). \tag{8.1.4}$$

Q.E.D.

It now follows that

$$\lim \ \arctan(x) = -\ \lim \ \arctan(-x) = -\frac{\pi}{2}. \tag{8.1.5}$$

$x \to -\infty$ $x \to -\infty$ z

Hence the range of the arctangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Proposition 8.1.4

If $x > 0$, then

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}. \tag{8.1.6}$$

Proof

Using the substitution $t = \frac{1}{u}$, we have

$$\begin{aligned} \arctan\left(\frac{1}{x}\right) &= \int_0^{\frac{1}{x}} \frac{1}{1+t^2} dt \\ &= \int_{+\infty}^{x} \frac{1}{1+\frac{1}{u^2}} \left(-\frac{1}{u^2}\right) du \\ &= -\int_{+\infty}^{x} \frac{1}{1+u^2} du \\ &= \int_{x}^{+\infty} \frac{1}{1+u^2} du \\ &= \frac{\pi}{2} - \int_0^x \frac{1}{1+u^2} du \\ &= \frac{\pi}{2} - \arctan(x). \end{aligned} \tag{8.1.7}$$

Q.E.D.

Proposition 8.1.5

If $x < 0$, then

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}. \tag{8.1.8}$$

Proof

The result follows immediately from the preceding proposition and the fact that arctangent is an odd function.

Exercise 8.1.1

Show that $\arctan(1) = \frac{\pi}{4}$ and $\arctan(-1) = -\frac{\pi}{4}$.

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8.2: The Tangent Function

Let

$$A = \left\{ \frac{\pi}{2} + n\pi : n \in \mathbb{Z} \right\} \tag{8.2.1}$$

and $D = \mathbb{R} \backslash A$. Let

$$t : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R} \tag{8.2.2}$$

be the inverse of the arctangent function. Note that t is increasing and differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We may extend t to a function on D as follows: For any $x \in D$, let

$$g(x) = \sup \left\{ n : n \in \mathbb{Z}, -\frac{\pi}{2} + n\pi < x \right\} \tag{8.2.3}$$

and define $T(x) = t(x - g(x)\pi)$.

Definition

With the notation of the above discussion, for any $x \in D$, we call the value $T(x)$ the tangent of x, which we denote $\tan(x)$.

Proposition 8.2.1

The tangent function has domain D (as defined above), range $\mathbb{R}$, and is differentiable at every point $x \in D$. Moreover, the tangent function is increasing on each interval of the form

$$\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi \right), \tag{8.2.4}$$

$n \in \mathbb{Z}$, with

$$\tan\left(\left(\frac{\pi}{2} + n\pi \right) + \right) = -\infty \tag{8.2.5}$$

and

$$\tan\left(\left(\frac{\pi}{2} + n\pi \right) - \right) = +\infty. \tag{8.2.6}$$

Proof

These results follow immediately from our definitions. Q.E.D.

Definition

Let $E \subset \mathbb{R}$. We say a function $f : E \to \mathbb{R}$ is periodic if there exists a real number $p > 0$ such that, for each $x \in E, x + p \in E$ and $f(x + p) = f(x)$. We say p is the period of a periodic function f if p is the smallest positive number for which $f(x + p) = f(x)$ for all $x \in E$.

Proposition 8.2.2

The tangent function has period π.

Proof

The result follows immediately from our definitions. Q.E.D.

Proposition 8.2.3

(Addition formula for tangent)

For any $x, y \in D$ with $x + y \in D$,

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}. \tag{8.2.7}$$

Proof

Suppose $y_1, y_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $y_1 + y_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $x_1 = \tan(y_1)$ and $x_2 = \tan(y_2)$. Note that if $x_1 > 0$, then $x_1 x_2 \geq 1$ would imply that

$$x_2 \geq \frac{1}{x_1}, \tag{8.2.8}$$

which in turn implies that

$$\begin{aligned} y_1 + y_2 &= \arctan(x_1) + \arctan(x_2) \\ &\geq \arctan(x_1) + \arctan\left(\frac{1}{x_1}\right) \\ &= \frac{\pi}{2}, \end{aligned} \tag{8.2.9}$$

contrary to our assumptions. Similarly, if $x_1 < 0$, then $x_1 x_2 \geq 1$ would imply that

$$x_2 \leq \frac{1}{x_1}, \tag{8.2.10}$$

which in turn implies that

$$\begin{aligned} y_1 + y_2 &= \arctan(x_1) + \arctan(x_2) \\ &\leq \arctan(x_1) + \arctan\left(\frac{1}{x_1}\right) \\ &= -\frac{\pi}{2}, \end{aligned} \tag{8.2.11}$$

contrary to our assumptions. Thus we must have $x_1 x_2 < 1$. Moreover, suppose u is a number between $-x_1$ and x_2. If $x_1 > 0$, then

$$x_2 < \frac{1}{x_1}, \tag{8.2.12}$$

and so

$$u < \frac{1}{x_1}. \tag{8.2.13}$$

If $x_1 < 0$, then

$$x_2 > \frac{1}{x_1}, \tag{8.2.14}$$

and so

$$u > \frac{1}{x_1}. \tag{8.2.15}$$

Now let

$$x = \frac{x_1 + x_2}{1 - x_1 x_2}. \tag{8.2.16}$$

We want to show that

$$\arctan(x) = \arctan(x_1) + \arctan(x_2), \tag{8.2.17}$$

which will imply that

$$\frac{\tan(y_1)+\tan(y_2)}{1-\tan(y_1)\tan(y_2)}=\tan(y_1+y_2). \tag{8.2.18}$$

We need to compute

$$\arctan(x)=\arctan\left(\frac{x_1+x_2}{1-x_1x_2}\right)=\int_0^{\frac{x_1+x_2}{1-x_1x_2}}\frac{1}{1+t^2}dt. \tag{8.2.19}$$

Let

$$t=\varphi(u)=\frac{x_1+u}{1-x_1u}, \tag{8.2.20}$$

where u varies between $-x_1$, where $t=0$, and x_2, where $t=x$. Now

$$\varphi'(u)=\frac{(1-x_1u)-(x_1+u)(-x_1)}{(1-x_1u)^2}=\frac{1+x_1^2}{(1-x_1u)^2}, \tag{8.2.21}$$

which is always positive, thus showing that φ is an increasing function, and

$$\begin{aligned}\frac{1}{1+t^2}&=\frac{1}{1+\left(\frac{x_1+u}{1-x_1u}\right)^2}\\&=\frac{(1-x_1u)^2}{(1-x_1u)^2+(x_1+u)^2}\\&=\frac{(1-x_1u)^2}{(1+x_1^2)(1+u^2)}.\end{aligned} \tag{8.2.22}$$

Hence

$$\begin{aligned}\arctan(x)&=\int_{-x_1}^{x_2}\frac{1}{1+u^2}du\\&=\int_{-x_1}^{0}\frac{1}{1+u^2}du+\int_0^{x_2}\frac{1}{1+u^2}du\\&=-\int_0^{-x_1}\frac{1}{1+u^2}du+\arctan(x_2)\\&=-\arctan(-x_1)+\arctan(x_2)\\&=\arctan(x_1)+\arctan(x_2).\end{aligned} \tag{8.2.23}$$

Now suppose $y_1,y_2\in(-\frac{\pi}{2},\frac{\pi}{2})$ with $y_1+y_2>\frac{\pi}{2}$. Then $y_1+y_2\in(\frac{\pi}{2},\pi)$,$x_1>0,x_2>0$, and

$$x_2>\frac{1}{x_1}. \tag{8.2.24}$$

With u and x as above, note then that as u increases from $-x_1$ to $\frac{1}{x_1}$, t increases from 0 to $+\infty$, and as u increases from $\frac{1}{x_1}$ to x_2, t increases from $-\infty$ to x.

Hence we have

$$\begin{aligned}\arctan(x)+\pi&=\int_0^x\frac{1}{1+t^2}dt+\int_{-\infty}^0\frac{1}{1+t^2}dt+\int_0^{+\infty}\frac{1}{1+t^2}dt\\&=\int_{-\infty}^x\frac{1}{1+t^2}dt+\int_0^{+\infty}\frac{1}{1+t^2}dt\\&=\int_{\frac{1}{x_1}}^{x_2}\frac{1}{1+u^2}du+\int_{-x_1}^{\frac{1}{x_1}}\frac{1}{1+u^2}du\\&=\int_{-x_1}^{x_2}\frac{1}{1+u^2}du\\&=\arctan(x_2)-\arctan(-x_1)\\&=\arctan(x_2)+\arctan(x_1).\end{aligned} \tag{8.2.25}$$

Hence

$$\begin{aligned}\tan(y_1+y_2)&=\tan(y_1+y_2-\pi)\\&=\tan(\arctan(x))\end{aligned}$$

$$= \frac{x_1 + x_2}{1 - x_1 x_2} \tag{8.2.26}$$
$$= \frac{\tan(y_1) + \tan(y_2)}{1 - \tan(y_1)\tan(y_2)}.$$

The case when $x_1 < 0$ may be handled similarly; it then follows that the addition formula holds for all $y_1, y_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The case for arbitrary $y_1, y_2 \in D$ with $y_1 + y_2 \in D$ then follows from the periodicity of the tangent function. Q.E.D.

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8.3: The Sine and Cosine Functions

We begin by defining functions

$$s : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R} \tag{8.3.1}$$

and

$$c : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R} \tag{8.3.2}$$

by

$$s(x) = \begin{cases} \frac{\tan(x)}{\sqrt{1+\tan^2(x)}}, & \text{if } x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ 1, & \text{if } x = \frac{\pi}{2} \end{cases} \tag{8.3.3}$$

and

$$c(x) = \begin{cases} \frac{1}{\sqrt{1+\tan^2(x)}}, & \text{if } x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ 0, & \text{if } x = \frac{\pi}{2}. \end{cases} \tag{8.3.4}$$

Note that

$$\lim_{x \to \frac{\pi}{2}-} s(x) = \lim_{y \to +\infty} \frac{y}{\sqrt{1+y^2}} = \lim_{y \to +\infty} \frac{1}{\sqrt{1+\frac{1}{y^2}}} = 1 \tag{8.3.5}$$

and

$$\lim_{x \to \frac{\pi}{2}-} c(x) = \lim_{y \to +\infty} \frac{1}{\sqrt{1+y^2}} = \lim_{y \to +\infty} \frac{\frac{1}{y}}{\sqrt{1+\frac{1}{y^2}}} = 0, \tag{8.3.6}$$

which shows that both s and c are continuous functions.

Next, we extend the definitions of s and c to functions

$$S : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R} \tag{8.3.7}$$

and

$$C : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R} \tag{8.3.8}$$

by defining

$$S(x) = \begin{cases} s(x), & \text{if } x \in (-\frac{\pi}{2}, \frac{\pi}{2}], \\ -s(x-\pi), & \text{if } x \in (\frac{\pi}{2}, \frac{3\pi}{2}] \end{cases} \tag{8.3.9}$$

and

$$C(x) = \begin{cases} c(x), & \text{if } x \in (-\frac{\pi}{2}, \frac{\pi}{2}], \\ -c(x-\pi), & \text{if } x \in (\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases} \tag{8.3.10}$$

Note that

$$\lim_{x \to \frac{\pi}{2}+} S(x) = \lim_{x \to -\frac{\pi}{2}+} -s(x) = -\lim_{y \to -\infty} \frac{y}{\sqrt{1+y^2}} = -\lim_{y \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{y^2}}} = 1 \tag{8.3.11}$$

and

$$\lim_{x \to \frac{\pi}{2}+} C(x) = \lim_{x \to -\frac{\pi}{2}+} -c(x) = -\lim_{y \to -\infty} \frac{1}{\sqrt{1+y^2}} = -\lim_{y \to -\infty} \frac{\frac{1}{y}}{-\sqrt{1+\frac{1}{y^2}}} = 0, \tag{8.3.12}$$

which shows that both S and C are continuous at $\frac{\pi}{2}$. Thus both S and C are continuous.

Finally, for any $x \in \mathbb{R}$, let

$$g(x) = \sup\left\{n : n \in \mathbb{Z}, -\frac{\pi}{2} + 2n\pi < x\right\} \tag{8.3.13}$$

and define

$$\sin(x) = S(x - 2\pi g(x)) \tag{8.3.14}$$

and

$$\cos(x) = C(x - 2\pi g(x)). \tag{8.3.15}$$

Definition

With the notation as above, for any $x \in \mathbb{R}$, we call $\sin(x)$ and $\cos(x)$ the sine and cosine of x, respectively.

Proposition 8.3.1

The sine and cosine functions are continuous on $\mathbb{R}$.

Proof

From the definitions, it is sufficient to verify continuity at $\frac{3\pi}{2}$. Now

$$\lim_{x \to \frac{3\pi}{2}-} \sin(x) = \lim_{x \to \frac{3\pi}{2}-} S(x) = S\left(\frac{3\pi}{2}\right) = -s\left(\frac{\pi}{2}\right) = -1 \tag{8.3.16}$$

and

$$\begin{aligned} \lim_{x \to \frac{3\pi}{2}+} \sin(x) &= \lim_{x \to \frac{3\pi}{2}+} S(x - 2\pi) \\ &= \lim_{x \to -\frac{\pi}{2}+} s(x) \\ &= \lim_{y \to -\infty} \frac{y}{\sqrt{1+y^2}} \\ &= \lim_{y \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{y^2}}} \\ &= -1, \end{aligned} \tag{8.3.17}$$

and so sine is continuous at $\frac{3\pi}{2}$. Similarly,

$$\lim_{x \to \frac{3\pi}{2}-} \cos(x) = \lim_{x \to \frac{3\pi}{2}-} C(x) = C\left(\frac{3\pi}{2}\right) = -c\left(\frac{\pi}{2}\right) = 0 \tag{8.3.18}$$

and

$$\begin{aligned} \lim_{x \to \frac{3\pi}{2}+} \cos(x) &= \lim_{x \to \frac{3\pi}{2}+} C(x - 2\pi) \\ &= \lim_{x \to -\frac{\pi}{2}+} c(x) \\ &= \lim_{y \to -\infty} \frac{1}{\sqrt{1+y^2}} \\ &= \lim_{y \to -\infty} \frac{\frac{1}{y}}{-\sqrt{1+\frac{1}{y^2}}} \\ &= 0, \end{aligned} \tag{8.3.19}$$

and so cosine is continuous at $\frac{3\pi}{2}$. Q.E.D.

8.3.1 Properties of sine and cosine

Proposition 8.3.2

The sine and cosine functions are periodic with period 2π.

Proof

The result follows immediately from the definitions. Q.E.D.

Proposition 8.3.3

For any $x \in \mathbb{R}, \sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

Proof

The result follows immediately from the definitions. Q.E.D.

Proposition 8.3.4

For any $x \in \mathbb{R}, \sin^2(x) + \cos^2(x) = 1$.

Proof

The result follows immediately from the definition of s and c. Q.E.D.

Proposition 8.3.5

The range of both the sine and cosine functions is $[-1, 1]$.

Proof

The result follows immediately from the definitions along with the facts that

$$\sqrt{1+y^2} \geq \sqrt{y^2} = |y| \tag{8.3.20}$$

and

$$\sqrt{1+y^2} \geq 1 \tag{8.3.21}$$

for any $y \in \mathbb{R}$. Q.E.D

Proposition 8.3.6

For any x in the domain of the tangent function,

$$\tan(x) = \frac{\sin(x)}{\cos(x)}. \tag{8.3.22}$$

Proof

The result follows immediately from the definitions. Q.E.D.

Proposition 8.3.7

For any x in the domain of the tangent function,

$$\sin^2(x) = \frac{\tan^2(x)}{1+\tan^2(x)} \tag{8.3.23}$$

and

$$\cos^2(x) = \frac{1}{1+\tan^2(x)}. \tag{8.3.24}$$

Proof

The result follows immediately from the definitions. Q.E.D.

Proposition 8.3.8

For any $x, y \in \mathbb{R}$,

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y). \tag{8.3.25}$$

Proof

First suppose x, y, and $x+y$ are in the domain of the tangent function.

Then

$$\begin{aligned}
\cos^2(x+y) &= \frac{1}{1+\tan^2(x+y)} \\
&= \frac{1}{1+\left(\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\right)^2} \\
&= \frac{(1-\tan(x)\tan(y))^2}{1-\tan(x)\tan(y))^2+(\tan(x)+\tan(y))^2} \\
&= \frac{(1-\tan(x)\tan(y))^2}{(1+\tan^2(x))(1+\tan^2(y))} \\
&= \left(\frac{1}{\sqrt{1+\tan^2(x)}\sqrt{1+\tan^2(y)}} - \frac{\tan(x)\tan(y)}{\sqrt{1+\tan^2(x)}\sqrt{1+\tan^2(y)}}\right)^2 \\
&= (\cos(x)\cos(y)-\sin(x)\sin(y))^2.
\end{aligned} \tag{8.3.26}$$

Hence

$$\cos(x+y) = \pm(\cos(x)\cos(y) - \sin(x)\sin(y)). \tag{8.3.27}$$

Consider a fixed value of x. Note that the positive sign must be chosen when $y = 0$. Moreover, increasing y by π changes the sign on both sides, so the positive sign must be chosen when y is any multiple of π. Since sine and cosine are continuous functions, the choice of sign could change only at points at which both sides are 0, but these points are separated by a distance of π, so we must always choose the positive sign. Hence we have

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{8.3.28}$$

for all $x, y \in \mathbb{R}$ for which x, y, and $x+y$ are in the domain of the tangent function. The identity for the other values of x and y now follows from the continuity of the sine and cosine functions. Q.E.D.

Proposition 8.3.9

For any $x, y \in \mathbb{R}$,

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x). \tag{8.3.29}$$

Exercise 8.3.1

Prove the previous proposition.

Exercise 8.3.2

Show that for any $x \in \mathbb{R}$,

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \tag{8.3.30}$$

and

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x). \tag{8.3.31}$$

Exercise 8.3.3

Show that for any $x \in \mathbb{R}$,

$$\sin(2x) = 2\sin(x)\cos(x) \tag{8.3.32}$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x). \tag{8.3.33}$$

Exercise 8.3.4

Show that for any $x \in \mathbb{R}$,

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \tag{8.3.34}$$

and

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}. \tag{8.3.35}$$

Exercise 8.3.5

Show that

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \tag{8.3.36}$$

$$\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \tag{8.3.37}$$

and

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}. \tag{8.3.38}$$

8.3.2 The calculus of the trigonometric functions

Proposition 8.3.10

$\lim_{x\to 0} \frac{\arctan(x)}{x} = 1.$

Proof

Using l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\arctan(x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1. \tag{8.3.39}$$

Q.E.D

Proposition 8.3.11

$\lim_{x\to 0}\frac{\tan(x)}{x}=1.$

Proof

Letting $x=\arctan(u)$, we have

$$\lim_{x\to 0}\frac{\tan(x)}{x}=\lim_{u\to 0}\frac{u}{\arctan(u)}=1. \tag{8.3.40}$$

Q.E.D

Proposition 8.3.12

$\lim_{x\to 0}\frac{\sin(x)}{x}=1.$

Proof

We have

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{\tan(x)}{x}\cos(x)=1. \tag{8.3.41}$$

Proposition 8.3.13

$\lim_{x\to 0}\frac{1-\cos(x)}{x}=0.$

Proof

We have

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{x}&=\lim_{x\to 0}\left(\frac{1-\cos(x)}{x}\right)\left(\frac{1+\cos(x)}{1+\cos(x)}\right)\\&=\lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))}\\&=\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(x)}{1+\cos(x)}\right)\\&=(1)(0)\\&=0.\end{aligned} \tag{8.3.42}$$

Q.E.D

Proposition 8.3.14

If $f(x)=\sin(x)$, then $f'(x)=\cos(x)$.

Proof

We have

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\&=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}\\&=\sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\&=\cos(x).\end{aligned} \tag{8.3.43}$$

Q.E.D

Proposition 8.3.15

If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Exercise 8.3.6

Prove the previous proposition.

Definition

For appropriate $x \in \mathbb{R}$, we call

$$\cot(x) = \frac{\cos x}{\sin(x)}, \tag{8.3.44}$$

$$\sec(x) = \frac{1}{\cos(x)}, \tag{8.3.45}$$

and

$$\csc(x) = \frac{1}{\sin(x)} \tag{8.3.46}$$

the cotangent, secant, and cosecant of x, respectively.

Exercise 8.3.7

If $f(x) = \tan(x)$ and $g(x) = \cot(x)$, show that

$$f'(x) = \sec^2(x) \tag{8.3.47}$$

and

$$g'(x) = -\csc^2(x). \tag{8.3.48}$$

Exercise 8.3.8

If $f(x) = \sec(x)$ and $g(x) = \csc(x)$, show that

$$f'(x) = \sec(x)\tan(x), \tag{8.3.49}$$

and

$$g'(x) = -\csc(x)\cot(x). \tag{8.3.50}$$

Proposition 8.3.16

$2\int_{-1}^{1}\sqrt{1-x^2}dx = \pi$.

Proof

Let $x = \sin(u)$. Then as u varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, x varies from -1 to 1. And, for these values, we have

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = |\cos(u)| = \cos(u). \tag{8.3.51}$$

Hence

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-x^2}dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2(u)du \\ &= \int_{-\frac{\pi}{2}}^{\frac{1}{2}}\frac{1+\cos(2u)}{2}du\end{aligned}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} du + \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2u) du \tag{8.3.52}$$
$$= \frac{\pi}{2} + \frac{1}{4}(\sin(\pi) - \sin(-\pi))$$
$$= \frac{\pi}{2}.$$

Q.E.D.

Exercise 8.3.9

Find the Taylor polynomial P_9 of order 9 for $f(x) = \sin(x)$ at 0. Note that this is equal to the Taylor polynomial of order 10 for f at 0. Is $P_9\left(\frac{1}{2}\right)$ an overestimate or an underestimate for $\sin\left(\frac{1}{2}\right)$? Find an upper bound for the error in this approximation.

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8.4: The Logarithm Functions

Definition

Given a positive real number x, we call

$$\log(x) = \int_1^x \frac{1}{t} dt \tag{8.4.1}$$

the logarithm of x.

Note that $\log(1) = 0, \log(x) < 0$ when $0 < x < 1$, and $\log(x) > 0$ when $x > 1$.

Proposition 8.4.1

The function $f(x) = \log(x)$ is an increasing, differentiable function with

$$f'(x) = \frac{1}{x} \tag{8.4.2}$$

for all $x > 0$.

Proof

Using the Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{1}{x} > 0 \tag{8.4.3}$$

for all $x > 0$, from which the result follows. Q.E.D.

Proposition 8.4.2

For any $x > 0$,

$$\log\left(\frac{1}{x}\right) = -\log(x). \tag{8.4.4}$$

Proof

Using the substitution $t = \frac{1}{u}$, we have

$$\log\left(\frac{1}{x}\right) = \int_1^{\frac{1}{x}} \frac{1}{t} dt = \int_1^x u\left(-\frac{1}{u^2}\right) du = -\int_1^x \frac{1}{u} du = -\log(x). \tag{8.4.5}$$

Q.E.D.

Proposition 8.4.3

For any positive real numbers x and y,

$$\log(xy) = \log(x) + \log(y). \tag{8.4.6}$$

Proof

Using the substitution $t = xu$, we have

$$\begin{aligned} \log(xy) &= \int_1^{xy} \frac{1}{t} dt \\ &= \int_{\frac{1}{x}}^{y} \frac{x}{xu} du \\ &= \int^1 \frac{1}{} du + \int^y \frac{1}{} du \end{aligned}$$

$$\begin{aligned} &= \int_{\frac{1}{x}}^{u} \frac{1}{u}du + \int_{1}^{y} \frac{1}{u}du \\ &= -\int_{1}^{\frac{1}{x}} \frac{1}{u}du + \log(y) \\ &= -\log\left(\frac{1}{x}\right) + \log(y) \\ &= \log(x) + \log(y). \end{aligned} \tag{8.4.7}$$

Q.E.D.

Proposition 8.4.4

If $r \in \mathbb{Q}$ and x is a positive real number, then

$$\log\left(x^r\right) = r\log(x). \tag{8.4.8}$$

Proof

Using the substitution $t = u^r$, we have

$$\log\left(x^r\right) = \int_1^{x^r} \frac{1}{t}dt = \int_1^x \frac{ru^{r-1}}{u^r}du = r\int_1^x \frac{1}{u}du = r\log(x). \tag{8.4.9}$$

Q.E.D.

Proposition 8.4.5

$\lim_{x\to+\infty} \log(x) = +\infty$ and $\lim_{x\to 0+} \log(x) = -\infty$.

Proof

Given a real number M, choose an integer n for which $n\log(2) > M$ (there exists such an n since $\log(2) > 0$). Then for any $x > 2^n$, we have

$$\log(x) > \log\left(2^n\right) = n\log(2) > M. \tag{8.4.10}$$

Hence $\lim_{x\to+\infty} \log(x) = +\infty$.

Similarly, given any real number M, we may choose an integer n for which $-n\log(2) < M$. Then for any $0 < x < \frac{1}{2^n}$, we have

$$\log(x) < \log\left(\frac{1}{2^n}\right) = -n\log(2) < M. \tag{8.4.11}$$

Hence $\lim_{x\to 0+} \log(x) = -\infty$. Q.E.D.

Note that the logarithm function has domain $(0, +\infty)$ and range $(-\infty, +\infty)$.

Exercise 8.4.1

Show that for any rational number $\alpha > 0$,

$$\lim_{x\to+\infty} x^\alpha = +\infty. \tag{8.4.12}$$

Proposition 8.4.6

For any rational number $\alpha > 0$,

$$\lim_{x\to+\infty} \frac{\log(x)}{x^\alpha} = 0. \tag{8.4.13}$$

Proof

Choose a rational number β such that $0 < \beta < \alpha$. Now for any $t > 1$,

$$\frac{1}{t} < \frac{1}{t} t^{\beta} = \frac{1}{t^{1-\beta}}. \tag{8.4.14}$$

Hence

$$\log(x) = \int_1^x \frac{1}{t} dt < \int_1^x \frac{1}{t^{1-\beta}} dt = \frac{x^{\beta} - 1}{\beta} < \frac{x^{\beta}}{\beta} \tag{8.4.15}$$

whenever $x > 1$. Thus

$$0 < \frac{\log(x)}{x^{\alpha}} < \frac{1}{\beta x^{\alpha-\beta}} \tag{8.4.16}$$

for $x > 1$. But

$$\lim_{x \to +\infty} \frac{1}{\beta x^{\alpha-\beta}} = 0, \tag{8.4.17}$$

so

$$\lim_{x \to +\infty} \frac{\log(x)}{x^{\alpha}} = 0. \tag{8.4.18}$$

Q.E.D.

? Exercise 8.4.2

Show that

$$\lim_{x \to 0^+} x^{\alpha} \log(x) = 0 \tag{8.4.19}$$

for any rational number $\alpha > 0$.

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8.5: The Exponential Function

Definition

We call the inverse of the logarithm function the exponential function. We denote the value of the exponential function at a real number x by $\exp(x)$.

Proposition 8.5.1

The exponential function has domain R and range $(0,+\infty)$. Moreover, the exponential function is increasing and differentiable on $\mathbb{R}$. If $f(x)=\exp(x)$, then $f'(x)=\exp(x)$.

Proof

Only the final statement of the proposition requires proof. If we let $g(x)=\log(x)$, then

$$f'(x)=\frac{1}{g'(\exp(x))}=\exp(x). \tag{8.5.1}$$

Q.E.D.

Proposition 8.5.2

For any real numbers x and y,

$$\exp(x+y)=\exp(x)\exp(y). \tag{8.5.2}$$

Proof

The result follows from

$$\log(\exp(x)\exp(y))=\log(\exp(x))+\log(\exp(y))=x+y. \tag{8.5.3}$$

Q.E.D.

Proposition 8.5.3

For any real number x,

$$\exp(-x)=\frac{1}{\exp(x)}. \tag{8.5.4}$$

Proof

The result follows from

$$\log\left(\frac{1}{\exp(x)}\right)=-\log(\exp(x))=-x. \tag{8.5.5}$$

Q.E.D.

Exercise 8.5.1

Use Thylor's theorem to show that

$$\exp(1)=e=\sum_{n=0}^{\infty}\frac{1}{n!}. \tag{8.5.6}$$

Proposition 8.5.4

For any rational number α,

$$\exp(\alpha) = e^{\alpha}. \tag{8.5.7}$$

Proof

Since $\log(e) = 1$, we have

$$\log\left(e^{\alpha}\right) = \alpha \log(e) = \alpha. \tag{8.5.8}$$

Q.E.D.

Definition

If α is an irrational number, we define

$$e^{\alpha} = \exp(\alpha). \tag{8.5.9}$$

Note that for any real numbers x and y,

$$e^{x+y} = e^{x} e^{y} \tag{8.5.10}$$

and

$$e^{-x} = \frac{1}{e^{x}}. \tag{8.5.11}$$

Moreover, $\log\left(e^{x}\right) = x$ and, if $x > 0, e^{\log(x)} = x$.

Definition

If x and a are real numbers with $a > 0$, we define

$$a^{x} = e^{x \log(a)}. \tag{8.5.12}$$

Exercise 8.5.2

Define $f : (0, +\infty) \to \mathbb{R}$ by $f(x) = x^{a}$, where $a \in \mathbb{R}, a \neq 0$. Show that $f'(x) = a x^{a-1}$.

Exercise 8.5.3

Suppose a is a positive real number and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = a^{x}$. Show that $f'(x) = a^{x} \log(a)$.

Proposition 8.5.5

For any real number $\alpha > 0$,

$$\lim_{x \to +\infty} x^{\alpha} e^{-x} = 0. \tag{8.5.13}$$

Proof

We know that

$$\lim_{y \to +\infty} \frac{\log(y)}{y^{\frac{1}{2}}} = 0. \tag{8.5.14}$$

Hence

$$\lim_{y \to +\infty} \frac{(\log(y))^{\alpha}}{y} = 0. \tag{8.5.15}$$

Letting $y = e^x$, we have

$$\lim_{x\to+\infty} \frac{x^\alpha}{e^x} = 0. \tag{8.5.16}$$

Q.E.D.

Proposition 8.5.6

For any real number α,

$$\lim_{x\to+\infty} \left(1 + \frac{\alpha}{x}\right)^x = e^\alpha. \tag{8.5.17}$$

Proof

First note that, letting $x = \frac{1}{h}$,

$$\lim_{x\to+\infty} \left(1 + \frac{\alpha}{x}\right)^x = \lim_{h\to 0+} (1+\alpha h)^{\frac{1}{h}} = \lim_{h\to 0^+} e^{\frac{1}{h}\log(1+\alpha h)}. \tag{8.5.18}$$

Using I'Hópital's rule, we have

$$\lim_{h\to 0^+} \frac{\log(1+\alpha h)}{h} = \lim_{h\to 0^+} \frac{\alpha}{1+\alpha h} = \alpha, \tag{8.5.19}$$

and the result follows from the continuity of the exponential function. Q.E.D.

Definition

We define the hyperbolic sine and hyperbolic cosine functions by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{8.5.20}$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{8.5.21}$$

respectively.

Exercise 8.5.4

Show that for any real numbers x and y,

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x) \tag{8.5.22}$$

and

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y). \tag{8.5.23}$$

Exercise 8.5.5

Show that for any real number x,

$$\cosh^2(x) - \sinh^2(x) = 1. \tag{8.5.24}$$

Exercise 8.5.6

If $f(x) = \sinh(x)$ and $g(x) = \cosh(x)$, show that

$$f'(x) = \cosh(x) \tag{8.5.25}$$

and

$$g'(x) = \sinh(x). \tag{8.5.26}$$

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